The Global World of A. Dermanis and an attempt to use System Dynamics for the analysis of Polar Motion (POM) and Length of Day Variations (LOD)

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0. A. Dermanis, his works, and the system theory of Polar Motion (POM) and Length of Day Variations (LOD), an introduction

At first we review the very important works of A. Dermains with respect to *Geodetic Reference Frames*, based on *surface deformation measures* like *dilatation, shear, rotation and energy*. Special attention finds *Frame Invariance and Parameter Estimability*.

The following parts of the author's work on Global Reference Frames in concentrated on the subject of System Theory for Polar Motion (POM) and Length of Day Variations (LOD) experienced by studying Bulletin A, Vol. XXIX, No. 046, on November 17, 2016. The analysis the new USNO VLBI solution using
least Earth Orientation Parameters (EOP). The contributed analysis of results is based on data from

- Very Long Baseline Interferometry (VLBI),
- Satellite Laser Ranging (SLR),
- Global Positioning System (GPS) satellites,
- Lunar Laser Ranging (LLR), and
- meteorological predictions for variations in Atmosphere Angular Momentum (AAM)

They start with the International Earth Rotation System (IERS) Rapid Service for weekly outputs, predictions of the Polar Motion as well as the difference between UT1-UTC (Universal Time Coordinated), also daily and Celestial Pole Ofsets Series.

1. Athanasios Dermanis and the problem of Geodetic Reference Frames, an introduction

Athanasios Dermanis, in short "Sakis", pioneered the topic of Geodetic Reference Frame by studying Earth Rotation and Network Geometry by studying the optimization of Very Long Baseline Interferometry (1977, 1978i, ii, 1980). Our first joint paper published in the prestigious "Geophysical Journal of the Royal Astronomical

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Society 64(1981)31-56" treated the estimability problem of geodetic, astronomical and geodynamical quantified in "Very-Long-Baseline-Interferometry" honoring the 90th birthday of Sir Harold Jeffarys. My own contribution was oriented to include relativistic terms which had been beforehand declared as measuring errors on the post-Newton level.

Sakis widened his interest in the analysis dilation, shear, rotation and energy for deformable, rotating bodies. (1982, 1983i, ii, iii, 1984) We wrote another joint paper by the finite element approach to the geodetic computation of two- and three-dimensional deformation parameters, a study of Frame Invariance and Parameter Estimability in the year 1992.

Earth rotation became for him of central importance and was the subject of key interest in Geodetic Reference Frame, in general. (2000; 2001; 2003i; ii; iii; 2005i; ii; 2006; 2007; 2009i; ii; 2010i; ii): *Fundamentals of Surface deformation* and application to *construction monitoring* were key subjects in (2011, 2013). Global frames of reference as well as determination of transformation parameters between various geodetic frames of reference are of key importance for him (2015, 2016i, ii).

Sakis is for many years the key theoretician in analysis systematically time deforming geodetic network in one-, two-, three- and four-dimensions! He gave advice to colleagues for the analysis of Global Reference Frames and the transformation between them! We hope that he likes my also systematic study of Polar Motion (POM) and Length of Day Variations (LOD) by means of System Theory.

2. System theory: The dynamical Euler-Liouville equations-angular momentum balance, and excitation functions

2.1 Introductory remarks: angular momentum balance, rotation axis, axis of figure

In order to develop a System Theory of the rotation of celestial bodies we have to take advantage of the rotational motion on the basis of the balance equations of linear and angular momentum of a deformable body. We have to refer the basic work of M.L. Smith (1974,1981), P. Georghiadun (1984) and P. Gorghiadun and E. Grafarend (1986). It is well known that the balance equations of the angular momentum are sufficient to describe the rotational motion of a rigid body, but not so far a deformable body. The equations of linear and angular momentum for a deformable body are coupled. Excitation mechanisms generate torques due to tidal effects by the Sun, the Moon and other planets for instance. We are in need to develop multibody dynamics of deformable bodies! Beside external acting momentums there are internal effects which are effecting torques, for instance loading mechanisms to the mantle caused by the triple system oceans atmosphere-solid celestial body, transversal surface stress and various core mantle interactions effects of first order. At this point we must mention the basic work of C. Truesdell (1961): According to his table, we must neglect here spin angular momentum, the momentum stress as well as degree of freedom of type Cosserat Continuum which relate to the antisymmetric part of the stress tensor.

2.2 Dynamical reference frames and rigid body dynamics

We start from here the commutative diagrams in decomposing the velocity field in three parts, namely in *Figure* 2.1 and *Figure* 2.2 and 2.1 and *Table* 2.2. L_o L_1 and L_2 . The balance of the moment of the momentum in a quasi-body fixed frame of reference is organizing in the *spin-orbiting coupling* L_o , and in the rotation-deformation coupling $L_1 + L_2 = L + \delta L$ which L_1 accounts for the torque of the rigid body as a first approximation and L_2 for the torque cancel by the *dynamics of deformable bodies of Liouville type* as *a second order approximation*.

In our perturbed balance equation we have denoted by w_i or $\{w_1, w_2, w_3\}$ the anholonomic three coordinates of the rotation vector in the quasi-inertial reference frame, $\{f_1, f_2, f_3 \mid 0\}$ an orthonormal Frenet tried, ξ_{kl} the Cartesian coordinates of the inertial tensor

$$\sum_{k,l=1}^{3} f^k \otimes f^l j_{kl} \tag{1}$$

in the frame of reference $\{f_1, f_2, f_3 \mid 0\}$ with respect to the mass center 0 of the planet. Since the body of celestial mechanics deforms, the coordinates j_{kl} of the tensor of inertial are time dependent, contrary to the rigid body dynamics, namely $dj_{kl} / dt \neq 0$. While

$$L_k = j_{kl}\omega_l^{\bullet} + j_{kl}^{\bullet}\omega_l + \delta_{ijk} j_{jm}\omega_i\omega_m \tag{2}$$

are the coordinates of the reference angular momentum, the so-called "incremental angular momentum" in the quasi-body fixed frame of reference $\{f_1, f_2, f_3 \mid 0\}$, we refer the perturbed angular momentum, the "incremental angular momentum" by

$$\delta L_k = \delta_{ijk}\omega_i \, \delta L_j \tag{3}$$

we introduce the Liouville perturbation theory based on L. Euler in E. Grafarend and K. H. Haner (1976) including second terms. The acting moments

$$\sum_{k=1}^{3} f^k \delta m_k \tag{4}$$

are placed on the right side of the balance equations, δj_{kl} denotes the incremental moments of inertia usually on the left side of the balance equations. The structure of the balance equations of angular momentum is generated by the triple decomposition of the local velocity field in Table 2.1. The term of zero order is determined by the velocity υ, for instance of the center of mass of the Earth, obtained by "COM", relative to the inertial reference point 0. Illustrated by Figure 2.2.

The contribution of the first order of the velocity field vi results in a velocity field generated by the rotational dynamics, the second order velocity field by the time dependent displacement field of the deformable body relative to the global rotation". The term v2 is caused by the "relative angular momentum" $L-2$, also called δL.

The term of zero order is related to the "spin-orbit coupling" related later. The other coupling terms of first and second order are mentioned by the "rotational deformation coupling", again treated separately.

While in Table 2.2 we have introduced the linearization of the rotational deformation, also called *"first Liouville perturbation"*, we used in addition the linearization of the tensor of inertia $J = j + \delta j$ and of the rotation vector $\Omega = \omega + \delta\omega$ also called the *"second Liouville perturbation"*. Also, the second perturbation applies to the torque $M = m + \delta m$. The splitting of the balance equations of angular momentum leads in the first approximation to the classical
rigid body rotation and in the second approximation to *the incremental angular momentum balanced with terms of second order*. In order to present simple solution of the incremental angular momentum equations we agree to two assumptions:

First, we fix the axis of the reference inertial tensor, the so-called eigenvectors, to the principle components of the inertial tensor, namely

$$j_{11} = A^*, j_{22} = B^*, j_{33} = C^*$$

assuming $D^* = E^* = F^* = 0$.

Second, we assume the reference rotation follows the so-called z-axis. In *system dynamics* upped *two balance equations of polar motion* with *dimensionless parameters* $x_1 := \delta\omega_1 / \omega$, $x_2 := \delta\omega_2 / \omega$! But, in the balance equation of the perturbation of the *Length of Day* (LOD) the dimensionless perturbation parameter $x_3 := \delta\omega_3 / \omega$ is introduced decoupled from the other two components (x_1, x_2).

2.3 The Euler balance equation of angular momentum: Liouville perturbation theory

The next step of the *Liouville's perturbation theory of angular momentum* is directed towards modeling the timelike variations of the incremental inertia tensor δj_{kl}. These solutions have been developed with respect to the local linear momentum balance equations, for instance solving *the gravitoviscoelastic field equations* in the *Habilitation Thesis* of D. Wolf (1997). We introduced the solutions in time-varying incremental inertial tensor in E. Grafarend, J. Engels and P. Varga (2000).

Here again refer to a recouping of the incremental inertia tensor towards the incremental potential coefficients $\delta\omega$ of degree 2, $m = \{-2, -1, 0, +1, +2\}$ specified to

- loading effect in the time domain, retarded
- tidal effects in the time domain, retarded
- centrifugal potential in the time domain, retarded

referring to the *Love number* $k_{2,R}$ or to the *Love kernel number function* $k_R(t-t')$ *International Reference Sphere* of radius *R*. Specifically, the *"Fluid Love Number"* $k_{2,f}$ are reviewed. In particular, the influence f of the incremental centrifugal potential was studied since it is linear in $\{x, x^{\bullet}\}$.

Table 2.1: Principle of Balance of moment of Momentum: angular momentum in a quasi-body fixed frame of reference (rotation reference frame) first Liouville perturbation

"three constituents of angular momentum"

$$L_o + L_1 + L_2$$

$$L_1 + L_2 = L + \delta L$$

"spin orbit coupling'

$$D_t L_o + \Omega \times L_o = M_o$$

$$L_o := M(\upsilon_o \times x_o)$$

"orbit angular momentum"
"rotational-deformation coupling"

$$D_t(L_1 + L_2) + \Omega \times (L_1 + L_2) = M_1 + M_2$$

$$J_{kl} D_t \Omega_l + (D_t J_{kl})\Omega_l + \delta_{ijk}\Omega_i J_{jm}\Omega_m + D_t \delta L_k + \delta_{ijk}\Omega_i \delta L_j = M_k$$

"inertia tensor"

$$J = f^k \otimes f^l \mathbf{J}_{kl}$$

(summation convention over repeated indicies)

$$\mathbf{J}_{kl} := \oiiint \rho(x,y,z)[\| x \|^2 \ \delta_{kl} - x_k x_l]\, dx\, dy\, dz$$

"angular momentum"

$$L_1 := \oiiint \rho(x,y,z)[\upsilon_1(x) \times x\, \delta_{kl} - x_k x_l]\, dx\, dy\, dz$$

$$L_2 := \oiiint \rho(x,y,z)[\upsilon_2(x) \times x\, \delta_{kl} - x_k x_l]\, dx\, dy\, dz$$

End of Table 2.1: first Liouville perturbation

Table 2.2: Fundamental decompositions of the velocity fields

$$\upsilon(x,t) = \upsilon_o(x,t) + \upsilon_2(x,t)$$

zero order velocity $\mathbf{v}_o$

the **zero-order velocity** $\mathbf{v}_o$ represents the velocity of the center of mass (COM) of the celestial body with respect to an inertially moving reference center

first order velocity $\mathbf{v}_1$

the **first order velocity** $\mathbf{v}_1$ represents the rotational velocity of type

$$rot \upsilon_1 = 2\omega .$$

Corollary

$$\mathbf{v}_1 = \omega \times \mathbf{x}$$

$$\upsilon_i = \delta_{ijk} \omega_j x_k$$

$$\mathbf{\Omega} = -\mathbf{\Omega}^T$$

second order velocity $\mathbf{v}_2$

the **second order velocity** $\mathbf{v}_2(x,t)$ represents the displacement rate of the deformable body (celestial body)

End of Table 2.2: Fundamental decompositions of the velocity fields

Figure 2.1: Decomposition of the velocity field $\mathbf{v}(x,t)$ commutation diagrams, Placement diagram P

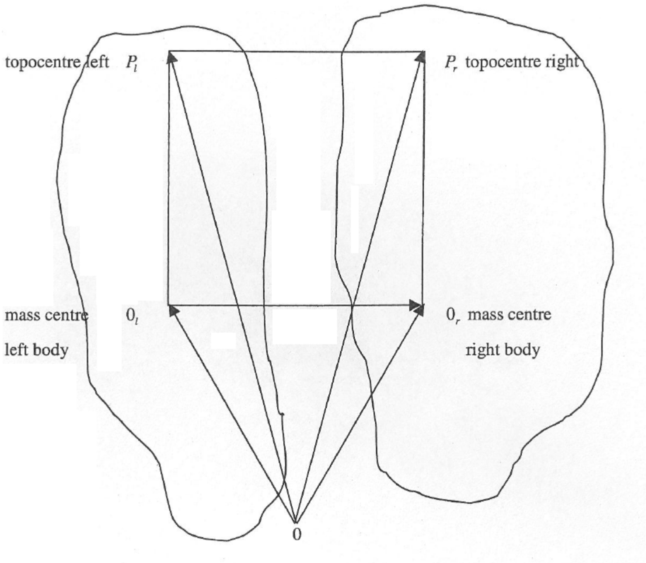


Figure 2.2: Frame of Reference: $\{E_1, E_2, E_3 \mid 0\}$ versus $\{f_1, f_2, f_3 \mid COM\}$, epochs t_1 and t_2, fixed frame *versus* moving frame

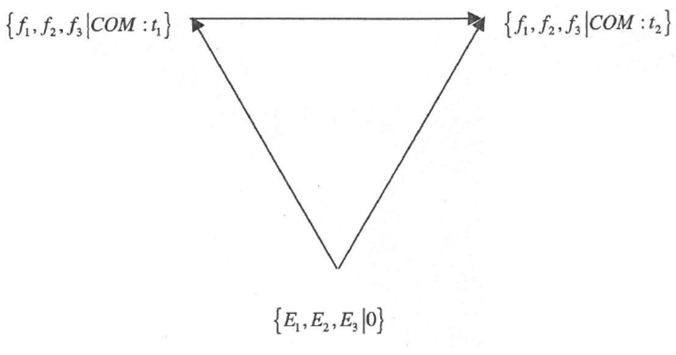


Figure 2.3: Principle of Balance of Moment of Momentum Angular Momentum in a quasi-body fixed frame of a reference Liouville perturbation theory

"inertial tensor"

$$J_{kl} = j_{kl} + \delta j_{kl}$$

special choice

$$j_{1,1} = A^*,\ j_{2,2} = B^*,\ j_{3,3} = C^*$$

(all other components vanish)

"rotation vector"

$$\Omega_k = \omega_k + \delta\omega_k$$

special choice: $\omega_3 = \omega$ *(all other components vanish)*

dimensionless x_l : $\delta\omega_k + \omega x_k$

"force moment torques"

$$M_k = m_k + \delta m_k$$

"Euler-Lioville equations of angular momentum"

$$\begin{aligned}(j_{kl} + \delta j_{kl}) D_t(\omega_k + \delta\omega_k) + (\omega_l + \delta\omega_l) D_t(j_{kl} + \delta j_{kl}) + \\ + \delta_{ijk}(\omega_i + \delta\omega_i)(j_{jm} + \delta j_{jk})(\omega_m + \delta\omega_m) + \\ + D_t\, \delta L_k + \delta_{ijk}(\omega_i + \delta\omega_j)\, \delta L_j = m_k + \delta m_k\end{aligned}$$

reference angular momentum equation

$$j_{kl} D_t \omega_l + \omega_l D_t j_{kl} + \delta_{ijk}\omega_i j_{jm}\omega_m = m_k$$

incremental angular momentum equation

$$j_{kl} D_t \delta\omega_l + \delta_{ijk} (\omega_m \delta\omega_i + \omega_i \delta\omega_m) +$$
$$+ \omega_l D_t j_{kl} + \delta_{ijk} \omega_i \omega_m \delta j_{jm} +$$
$$+ D_t \delta L_k + \delta_{ijk} \omega_i \delta L_j + 0(2) = \delta m_k$$

" $j_{1,1} \neq 0,\ j_{2,2} \neq 0,\ j_{3,3} \neq 0$: *all others vanish*"

$$1st: j_{1,1} D_t \delta\omega_1 + \omega_2 (j_{3,3} - j_{2,2}) \delta\omega_3 + \omega_3 (j_{3,3} - j_{2,2}) \delta\omega_2 + \omega_1 D_t j_{1,1} + \omega_2 D_t j_{1,2} + \omega_3 D_t j_{1,3} +$$
$$- \omega_3 \omega_1 \delta j_{2,1} + \omega_2 \omega_1 \delta j_{3,1} + \omega_2 \omega_3 (\delta j_{3,3} - \delta j_{2,2}) + \omega_2^2 \delta j_{3,1} - \omega_3^2 \delta j_{2,3} + D_t \delta L_1 + \omega_2 \delta L_3 - \omega_3 \delta L_2 + 0(2) = \delta m_1$$

$$2nd: j_{2,2} D_t \delta\omega_2 + \omega_3 (j_{1,1} - j_{3,3}) \delta\omega_1 + \omega_1 (j_{1,1} - j_{3,3}) \delta\omega_3 + \omega_2 D_t j_{2,2} + \omega_3 D_t j_{2,2} + \omega_1 D_t j_{2,1} +$$
$$- \omega_1 \omega_2 \delta j_{3,2} + \omega_3 \omega_2 \delta j_{1,2} + \omega_3 \omega_1 (\delta j_{1,1} - \delta j_{3,3}) + \omega_3^2 \delta j_{1,2} - \omega_1^2 \delta j_{3,1} + D_t \delta L_2 + \omega_3 \delta L_1 - \omega_1 \delta L_3 + 0(2) = \delta m_2$$

$$3rd: j_{3,3} D_t \delta\omega_3 + \omega_1 (j_{2,2} - j_{1,1}) \delta\omega_2 + \omega_2 (j_{2,2} - j_{1,1}) \delta\omega_1 + \omega_3 D_t j_{3,3} + \omega_1 D_t j_{3,1} + \omega_2 D_t j_{3,2} +$$
$$- \omega_2 \omega_3 \delta j_{1,3} + \omega_1 \omega_3 \delta j_{2,3} + \omega_1 \omega_2 (\delta j_{2,2} - \delta j_{1,1}) + \omega_1^2 \delta j_{2,3} - \omega_2^2 \delta j_{1,2} + D_t \delta L_3 + \omega_1 \delta L_2 - \omega_2 \delta L_1 + 0(2) = \delta m_3$$

$j_{1,1} = A^*,\ j_{2,2} = B^*,\ j_{3,3} = C^*$: all other j_{kl} vanish

and

$\omega_3 = \omega$: all other ω_i vanish

$$\omega x_1 := \delta\omega_1,\ \omega x_2 := \delta\omega_2,\ \omega x_3 := \delta\omega_3$$

$$1st: A\omega \dot{x}_1 + \omega^2 (C - B) x_2 + \omega \delta j_{1,3} - \omega^2 \delta j_{2,3} + \delta \dot{L}_1 - \omega \delta L_2 = x_1$$
$$2nd: B\omega \dot{x}_2 - \omega^2 (A - C) x_1 + \omega \delta j_{2,3} - \omega^2 \delta j_{3,1} + \delta \dot{L}_2 - \omega \delta L_1 = x_2$$
$$3rd: C\omega \dot{x}_3 + \omega \delta j_{3,3} + \delta \dot{L}_3 = x_3$$

End of Figure 2.3: Liouville equations

Example 2.1: Incremental inertia tensor generalized Mc Cullagh representation

$$\delta j_{1,3} = \delta i_{1,3} = -\int_0^{\pi} d\lambda \int_{-\pi/2}^{\pi/2} d\varphi \cos\varphi \int_0^R r^2 dr \delta\rho(\lambda, \varphi, r) xy$$

$$\delta j_{2,3} = \delta i_{2,3} = -\int_0^{\pi} d\lambda \int_{-\pi/2}^{\pi/2} d\varphi \cos\varphi \int_0^R r^2 dr \delta\rho(\lambda, \varphi, r) yz$$

$$\delta j_{3,3} = -\int_0^{\pi} d\lambda \int_{-\pi/2}^{\pi/2} d\varphi \cos\varphi \int_0^R r^2 dr \delta\rho(\lambda, \varphi, r)(x^2 + y^2)$$

"incremental gravitational potential (deformation potential)"

$$\delta u(\lambda,\varphi,r) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{R}{r}\right)^{l+1} e^{l,m}(\lambda,\varphi)\delta u_{l,m}$$

$$x = \frac{r}{\sqrt{3}}e^{1,1},\ \ y = \frac{r}{\sqrt{3}}e^{1,-1},\ \ z = \frac{r}{\sqrt{3}}e^{1,0}$$

"generalized Mc Cullagh representation"

$$\delta u_{2,1} = \sqrt{\frac{3}{5}}\frac{g}{R^3}\oiiint \delta\rho(x,y,z)xz\,dxdydz$$
$$\delta u_{2,-1} = \sqrt{\frac{3}{5}}\frac{g}{R^3}\oiiint \delta\rho(x,y,z)yz\,dxdydz$$

$$\delta j_{1,3} = \delta i_{1,3} = -\sqrt{\frac{5}{3}}\frac{R^3}{g}\delta u_{2,1},\ \delta j_{2,3} = \delta i_{2,3} = -\sqrt{\frac{5}{3}}\delta u_{2,-1},\ \delta j_{3,3} = -\frac{2\sqrt{5}}{3}\frac{R^3}{g}\delta u_{2,0}$$

End of Example 2.1: Incremental inertia

Example 2.2: Linearized centrifugal potential

$$\delta v(cent) = \langle\omega|\,\delta\omega\rangle x^2 - \langle\omega|\,x\rangle\langle\delta\omega|\,x\rangle$$

$$\delta v(cent) = \omega\left[\delta\omega_3(x^2+y^2+z^2) - \delta\omega_1 xz - \delta\omega_2 yz\right]$$

$$\delta v(cent) = \omega\left\{\delta\omega_3\frac{2}{3}r^2\left[1-\frac{1}{\sqrt{5}}e^{2,0}(\lambda,\varphi)\right] - \frac{r^2}{\sqrt{15}}\left[\delta\omega_1 e^{2,1}(\lambda,\varphi) + \delta\omega_2 e^{2,-1}(\lambda,\varphi)\right]\right\}$$
$$\delta v(cent) = \omega\,\delta\omega_3\frac{2}{3}r^2 + \frac{r^2}{R^2}\sum_{m=-1}^{1}\delta v_{2,m}(cent)\,e^{2,m}(\lambda,\varphi)$$

"coefficients of the lineared centrifugal potential"

$$\delta v_{2,1}(cent) = -\omega\frac{R^2}{\sqrt{15}}\delta\omega,\ \ \delta v_{2,0}(cent) = -\omega\frac{R^2}{\sqrt{15}}\delta\omega_1,\ \ \delta v_{2,-1}(cent) = -\omega\frac{R^2}{\sqrt{15}}\delta\omega_2$$

End of Example 2.2: Linearized centrifugal potential

Example 2.3: Love-Shida hypothesis, homogeneous spherical shell presented viscoelastic Earth model in the time domain (Earth radius R)

$$\delta\omega_{2,m}(t) = \left[1 + k_2(load, elastic)\right]\delta u_{2,m}(load) + \int_0^t k_{2,R}(load, t-t')\,\delta u_{2,m}(load, t')\,dt'$$

$$+ k_2(tidal, elastic)\,\delta u_{2,m}(tid, t) + \int_0^t k_{2,R}(tid, t-t')\,\delta u_{2,m}(tid, t')\,dt'$$

$$+ k_2(cent, elastic)\,\delta u_{2,m}(cent, t) + \int_0^t k_{2,R}(cent, t-t')\,\delta u_{2,m}(cent, t')(t')\,dt'$$

" k_2 (elastic) as a dimensionless constant: instantaneous reaction to the action of the excitation function"

" $k_{2,R}\,(t - t')$ is a Love viscoelastic kernel function on the terrestrial sphere S^2 of dimension 1/time. For a R homogeneous spherical shell viscoelastic Earth model, the Love kernel function $k_{2,R}\,(t - t')$ can be represented by

$$k_{2,R} = \sum_{j=1}^{J} k_j \exp(-s_j t)$$

J is the number of nodal points ("Nullstellen") of the secular determinant of the *Laplace transformed gravitoviscoelastic field equations*?"

"Fluid Love number"

The fluid Love number $k_{2,f}$ of degree 2 is achieved when we set the excitation function as a constant and if we go to the limit $t \to \infty$. In this way the model Earth has time to relax to the constant excitation.

$$k_{2,f} := k_2(elastic) + \lim_{t\to\infty} \int_0^t k_{2,R}(t-t')\,dt' =$$

$$= k_2(elastic) + \lim_{t\to\infty} \int_0^t \sum_{j-1}^{J} k_j \left[\exp - s_j(t-t')\right] dt' =$$

$$= k_2(elastic) + \sum_{j-1}^{J} \frac{k_j}{s_j}$$

End of Example 2.3: Love-Shida

Example 2.4: Parameter of an Earth model with 5 interfaces radius density shear modulus dynamic viscosity

Radius (m)	Density (kg/m^2)	Shear modulus (kg/ms^2)	Dynamic viscosity (kg/ms^2)
0000000.0			
	10932.0	$0.0000\times10^{+00}$	$0.0000\times10^{+00}$
3480000.0			
	4878.0	$0.2190\times10^{+12}$	$0.1000\times10^{+22}$
5701000.0			
	3857.0	$0.1060\times10^{+12}$	$0.1000\times10^{+22}$
5951000.0			
	3434.0	$0.7270\times10^{+11}$	$0.1000\times10^{+22}$
6250000.0			
	3184.0	$0.6020\times10^{+11}$	$0.1000\times10^{+25}$
6371000.0			

End of Example 2.4: Interfaces

Example 2.5: Nodal points of the secular equation number relaxiation time inverse relaxiation time

number	Relaxiation time (year)	Inverse relaxiation time (1/Jid)
1	250.9310	$-0.3985\times10^{+01}$
2	282.2692	$-0.3542\times10^{+01}$
3	352.8929	$-0.2833\times10^{+01}$
4	402.8933	$-0.2482\times10^{+01}$
5	494.2672	$-0.2023\times10^{+01}$
6	2224.9892	$-0.4494\times10^{+00}$
7	9083.7657	$-0.1100\times10^{+00}$
8	530740.5736	-0.1884×10^{-02}
9	708982.2371	-0.1410×10^{-02}
10	28063129.4519	-0.3563×10^{-04}
11	592709956.3718	-0.1687×10^{-05}

End of Example 2.5: Nodal points of the secular equation

Example 2.6: Load and Love number: components of various frequencies Load number Love number

number	Love number $K_{2,el}$(load) $0.3050\times10^{+00}$ $K_{2,R}$(load)	Love number $K_{2,el}$(tide, cent) $0.3050\times10^{+00}$ $K_{2,R}$(tide,cent)
1	$-0.1155\times10^{+00}$	$-0.1897\times10^{+00}$
2	-0.9956×10^{-01}	$-0.1136\times10^{+00}$
3	$-0.2743\times10^{+00}$	$-0.3671\times10^{+00}$
4	-0.7762×10^{-01}	-0.8120×10^{-01}
5	$-0.3350\times10^{+00}$	$-0.4265\times10^{+00}$
6	$-0.1409\times10^{+00}$	-0.8122×10^{-01}
7	-0.2190×10^{-03}	-0.8207×10^{-03}
8	-0.2396×10^{-05}	-0.2261×10^{-04}
9	-0.1077×10^{-03}	-0.1524×10^{-04}
10	-0.2219×10^{-06}	-0.1124×10^{-07}
11	-0.5612×10^{-08}	-0.1123×10^{-09}

End of Example 2.6: Load and Love number

Example 2.7: Incremental inertia tensor generated by the incremental centrifugal potential (generalized Mc Cullagh representation: J. Geodesy 74(2000), 519-530)

$$\gamma = {gm}/{R^2}$$

$$1st: \ \delta j_{1,3} = -\sqrt{\frac{5}{3}}\frac{R^3}{g}\delta u_{2,1} =$$

$$= \frac{\omega^2 R^5}{3g} k_2(cent, elastic) x_1(t) + \frac{\omega^2 R^5}{3g} \int_0^t k_2(t-t', cent) x_1(t')\,dt'$$

$$2nd: \ \delta j_{3,3} = -\sqrt{\frac{5}{3}}\frac{R^3}{g}\delta u_{2,-1}$$

$$= \frac{\omega^2 R^5}{3g} k_2(cent, elastic) x_2(t) + \frac{\omega^2 R^5}{3g} \int_0^t k_2(t-t', cent) x_2(t')dt'$$

$$3rd:\ \delta j_{3,3} = -\frac{2\sqrt{5}}{3}\frac{R^3}{g}\delta u_{2,0}$$

$$= \frac{4\omega^2 R^5}{9g} k_2(cent, elastic) x_3(t) + \frac{4\omega^2 R^5}{9g}\int_0^t k_2(t-t', cent) x_3(t') dt'$$

End of Example 2.7: incremental centrifugal potential

Example 2.8: Time derivative of the incremental inertia tensor generated by the incremental centrifugal potential

"3 terms"

$$1st:\ \dot{\delta j}_{1,3} = \frac{\omega^2 R^5}{3g}\left\{k_2(cent, elastic)\dot{x}_1(t) + k_{2,R}(0, cent) x_1(t) + \int_0^t \dot{k}_2(t-t', cent) x_1(t') dt'\right\}$$

$$2nd:\ \dot{\delta j}_{2,3} = \frac{\omega^2 R^5}{3g}\left\{k_2(cent, elastic)\dot{x}_2(t) + k_{2,R}(0, cent) x_2(t) + \int_0^t \dot{k}_2(t-t', cent) x_2(t') dt'\right\}$$

$$3rd:\ \dot{\delta j}_{3,3} = \frac{4\omega^2 R^5}{9g}\left\{k_2(cent, elastic)\dot{x}_3(t) + k_{2,R}(0, cent) x_3(t) + \int_0^t \dot{k}_2(t-t', cent) x_3(t') dt'\right\}$$

(R. P. Kanwal: Linear integral equation, Academic Press, New York — 1971 page 265, formula (2))

$$\frac{d}{dt}\int_{A(t)}^{B(t)} f(t,t')dt' = \int_A^B \frac{\partial f}{\partial t}(t,t')dt' + f\{t, B(t)\}\frac{dB}{dt} - f\{t, A(t)\}\frac{dA}{dt}$$

End of Example 2.8: Time derivative of the incremental centrifugal potential

2.3 Liouville perturbation theory: system equations in the time and in the Laplace-Fourier domain

The *Liouville perturbation theory* of the *Euler dynamical equations* of angular momentum of the Earth considered as a deformable body leads to a first order inhomogeneous system of integro-differential equations, which are classified in terms of system theory. With respect to a viscoelastic Earth model of homogeneous spherical shells the spectrum of the *Liouville operator* is analyzed. Following a proposal of *M. Schneider* (Proc. Bundesamt für Kartographie and Geodaesie 5, pp. 28-33, Frankfurt 1999) the first order system is differentiated to a second order system and being alternatively classified as a second order inhomogeneous system of integro-differential equations. It leads to the interpretation that the characteristic equations of *Polar Motion* represent an excited coupled, damped approximately elliptic oscillator while the characteristic equation of *Length-of Day* variation documents an excited, damped non-periodic motion. Solutions are represented both in the *Laplace domain* as well as in the *Fourier domain*. New solutions are presented in the dynamical waveled domain as well as in the fractal domain, tentatively.

It is one of the first contributions in applying techniques of *System Dynamics* when *M. Schneider* (1999) presented his variational equations for the study of *Polar Motion*: He moved the excitation functions of the *relative angular momentum*, namely the tidal effect, the leading terms and the core-mantle coupling, *for instance*, to the *right side* of the balance equations of angular momentum. They are effecting in line with the *incremental torques* the balance in a mathematical portray: they are called *"inhomogeneous part"*. We describe the basic equations in *Table 2.3* up to *Table 2.4* in terms of balance of momentum of momentum, namely angular momentum, for *Polar Motion* and *Length-of-Day* variation.

In detail, *Table 2.4* refers to the inergro-differential equations of type $X^{\bullet} = Ax + + f(x) + b$ of *Polar Motion*. As proposed by *M. Schneider* the first order differential equations were transformed into a system of *second order differential equations*. We identify in terms of a *second order differential equations Polar Motion* equations $X^{\bullet\bullet} + (F - A^2)x + (Af + f_{12}) = Ab + b^{\bullet}$,

as an excited damped approximately elliptic harmonic oscillator

In contrast, we analyze in the time domain the intergro-differential equation $x_3^{\bullet} = a_{33}x_3 + f_3(x_3) + b$ for Length of Day variation. The retarded system equation of first order are transformed to a system of second order $x_3^{\bullet\bullet} - a_3^2 x_3 - (a_{33}f_3 + f_3^{\bullet}) = = a_{33}b_3 + b_3^{\bullet}$, interpreted as an excited, damped, non-periodic

function due to $a_{33} > 0$.

At this end, we list some books on System Dynamics, for instance *A. M. O. Almeida* (1988), *M. W. Hirsch and S. Smale* (1974) and *L. Perko* (1996).

Table 2.3: Principle of Balance of momentum (angular momentum) in a quasi-body fixed frame of reference-polar motion equations in the time domain

"1st polar motion equation"

$$\left[A+\frac{\omega^2 R^5}{3g}k_2(cent,elastic)\right]\dot{x}_1+\frac{\omega^2 R^5}{3g}k_{2,R}(0,cent)x_1+\int_0^t \dot{k}_2(t-t',cent)x_1(t')dt'$$
$$+\omega\left[C-B-\frac{\omega^2 R^5}{3g}k_2(cent,elastic)\right]x_2-\frac{\omega^2 R^5}{3g}\int_0^t k_{2,R}(t-t',cent)x_2(t')dt'=$$
$$=f(incr,torque,rel.ang.mom.,tide,load,stress)$$

"2nd polar motion equation"

$$\left[B+\frac{\omega^2 R^5}{3g}k_2(cent,elastic)\right]\dot{x}_2+\frac{\omega^2 R^5}{3g}k_{2,R}(0,cent)x_1+\int_0^t \dot{k}_2(t-t',cent)x_2(t')dt'$$
$$+\omega\left[A-C-\frac{\omega^2 R^5}{3g}k_2(cent,elastic)\right]x_1-\frac{\omega^2 R^5}{3g}\int_0^t k_{2,R}(t-t',cent)x_1(t')dt'=$$
$$=g(incr,torque,rel.ang.mom.,tide,load,stress)$$

End of Table 2.3: Angular momentum, polar motion

Table 2.4: Polar motion equations in the time domain

"system of intergrow-differential equations of first order evolutionary equations"

$$1st:\dot{x}_1=\frac{\omega\left[B-C+\frac{\omega^2 R^5}{3g}k_2(cent,elastic)\right]x_2-\frac{\omega^2 R^5}{3g}k_{2,R}(0,cent)x_1-\int_0^t \dot{k}_2(t-t',cent)x_1(t')dt'+f}{\left[A+\frac{\omega^2 R^5}{3g}k_2(cent,elastic)\right]}$$

$$2nd:\dot{x}_2=\frac{\omega\left[A-C+\frac{\omega^2 R^5}{3g}k_2(cent,elastic)\right]x_1-\frac{\omega^2 R^5}{3g}k_{2,R}(0,cent)x_2-\int_0^t \dot{k}_2(t-t',cent)x_2(t')dt'+g}{\left[B+\frac{\omega^2 R^5}{3g}k_2(cent,elastic)\right]}$$

$$\boxed{\dot{x} = Ax + f(x) + b}$$

$$a_{1,1} = -\left[A + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \frac{\omega^2 R^5}{3g} k_{2,R}(0, cent)$$

$$a_{1,2} = -\left[A + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \omega \left[B - C + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]$$

$$a_{2,1} = -\left[B + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \omega \left[A - C + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]$$

$$a_{2,2} = -\left[B + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \frac{\omega^2 R^5}{3g} k_{2,R}(0, cent)$$

$$f_1(x) = -\left[A + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \int_0^t \dot{k}_2(t - t', cent) x_1(t') dt'$$

$$f_2(x) = -\left[B + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \int_0^t \dot{k}_2(t - t', cent) x_2(t') dt'$$

$$b_1 := \left[A + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} f \quad , \quad b_2 := \left[B + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} g$$

$$\begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} =: f(x), \quad \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} =: b$$

“the eigenvalues of the matrix A”

$$| A - \lambda I_2 | = 0 \Leftrightarrow \begin{vmatrix} a_{1,1} - \lambda & a_{1,2} \\ a_{2,1} & a_{2,2} - \lambda \end{vmatrix} = 0 \Leftrightarrow (a_{1,1} - \lambda)(a_{2,2} - \lambda) - a_{1,2} a_{1,1} = 0 \Leftrightarrow$$

$$\lambda^2 - \lambda(a_{1,1} + a_{2,2}) + a_{1,1} a_{2,2} - a_{1,2} a_{2,1} = 0 \Leftrightarrow \lambda^2 - \lambda trA + \det A = 0 \Leftrightarrow$$

$$\lambda_{1,2}(A) = \frac{1}{2} trA \pm \frac{1}{2} \sqrt{(trA)^2 - 4 \det A}$$

“1st invariant”

$$trA = a_{1,1} + a_{2,2} = -\left[A + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \frac{\omega^2 R^5}{3g} k_{2,R}(0, cent) +$$

$$-\left[B + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \frac{\omega^2 R^5}{3g} k_{2,R}(0, cent)$$

" 2nd invariant"

$$-\det A = a_{1,2}a_{2,1} - a_{1,1}a_{2,2} = -\left[A + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \left[B + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \times$$

$$\times\, \omega \left[B - C + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right] \omega \left[A - C + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right] +$$

$$+ \left[A + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \left[B + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \frac{\omega^2 R^5}{3g} k_{2,R}(0, cent)$$

"*special case*: $a_{1,1} = 0,\ a_{2,2} = 0,\ A = B,\ a_{1,2} = -a_{2,1}$"

$$\lambda_{1,2}(A) = \pm\sqrt{\det A} = \pm \left[A + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \omega \left[A - C + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]$$

End of Table 2.4: Polar motion, time domain

Table 2.5: Polar motion equations in the time domain of type second order

"system of intergro-differential equations of second order"

$$\dot{x} = Ax + f(x) + b$$

$$\ddot{x} = A\dot{x} + \dot{f} + \dot{b} = A(Ax + f(x) + b) + \dot{f} + \dot{b}$$

$$\ddot{x} = A^2 x + (Af + \dot{f}) + Ab + \dot{b}$$

$$\ddot{x} - A^2 x - (Af + \dot{f}) = Ab + \dot{b}$$

"*special case*: $a_{1,1} = 0,\ a_{2,2} = 0,\ A = B,\ a_{1,2} = -a_{2,1}$"

$$-A^2 = -\begin{bmatrix} a_{12}\, a_{21} & 0 \\ 0 & a_{12}\, a_{21} \end{bmatrix} = \begin{bmatrix} a_{12}^2 & 0 \\ 0 & a_{12}^2 \end{bmatrix}$$

$$\lambda_{1,2}(-A) = a_{1,2}^2 = \left[A^* + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-2} \omega \left[A^* - C^* + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^2 \in \mathbb{R}^+$$

"excited circular harmonic oscillator"

"general case: $a_{1,1} \neq 0$, $a_{2,2} \neq 0$, $A \neq B$, $a_{1,2} \neq -a_{2,1}$"

$$\lambda_1 \neq \lambda_2 \in \mathbb{R}^+$$

"excited elliptic harmonic oscillator"

$$f_1(x) = -\left[A^* + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \int_0^t \dot{k}_2(t-t', cent) x_1(t') dt'$$

$$f_2(x) = -\left[B^* + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \int_0^t \dot{k}_2(t-t', cent) x_2(t') dt'$$

$$\boxed{\begin{aligned} \dot{f}_1(x) &= f_{1,1} x_1 + \dot{f}_{1,2} \\ \dot{f}_2(x) &= f_{2,2} x_2 + \dot{f}_{2,2} \end{aligned}}$$

$$f_{1,1}(x) = -\left[A^* + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \dot{k}_2(0, cent)$$

$$\dot{f}_{1,2}(x) = -\left[A^* + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \int_0^t \ddot{k}_2(t-t', cent) x_1(t') dt'$$

$$f_{2,2}(x) = -\left[B^* + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \dot{k}_2(0, cent)$$

$$\dot{f}_{2,2}(x) = -\left[B^* + \frac{\omega^2 R^5}{3g} k_2(cent, elastic) \right]^{-1} \int_0^t \ddot{k}_2(t-t', cent) x_1(t') dt'$$

"system equations: system of integro differential equations of second order"

$$\ddot{x} - (F - A^2 x) - (Af + \dot{f}) = Ab + \dot{b}$$

$$F := \begin{bmatrix} f_{1,1} & 0 \\ 0 & f_{2,2} \end{bmatrix}$$

"excited nonlinear damped harmonic oscillator"

End of Table 2.5: Polar motion, time domain

Table 2.6: Principle of balance of Moment of Momentum (Angular Momentum) in a quasi-body fixed in frame if reference equation of Length of Day variation in the time domain

"intergro-differential equations of first order"

$$\left[C^* + \frac{4}{9}\frac{\omega^2 R^5}{g} k_2(cent, elastic) \right] \dot{x}_3 + \frac{4}{9}\frac{\omega^2 R^5}{g} k_{2,R}(0, cent) x_3 +$$

$$+ \int_0^t \dot{k}_2(t - t', cent) x_3(t') dt' + \frac{4}{9}\frac{\omega^2 R^5}{g} \int_0^t k_{2,R}(t - t', cent) x_3(t') dt' =$$

$$= h(incr.torque, rel.ang.mom., tide, load, stress)$$

"system of equations"

$$\dot{x}_3 = a_{3,3}\, x_3 + f_3(x_3) + b_3$$

$$a_{3,3} := \left[C^* + \frac{4}{9}\frac{\omega^2 R^5}{g} k_2(cent, elastic) \right]^{-1} \frac{4}{9}\frac{\omega^2 R^5}{g} k_{2,R}(0, cent) x_3$$

$$f_3(x_3) := \left[C^* + \frac{4}{9}\frac{\omega^2 R^5}{g} k_2(cent, elastic) \right]^{-1} \int_0^t \dot{k}_2(t - t', cent) x_3(t') dt'$$

$$b_3 := \left[C^* + \frac{4}{9}\frac{\omega^2 R^5}{g} k_2(cent, elastic) \right]^{-1} h$$

End of Table 2.6: Length of Day variation, time domain

Table 2.7: Length of Day variation in the time domain

"intergro-differential equations of second order"

$$\dot{x}_3 = a_{3,3}\, x_3 + f_3(x_3) + b_3$$

$$\ddot{x}_3 = a_{3,3}\dot{x}_3 + \dot{f}_3 + \dot{b}_3 = a_{3,3}(a_{3,3}x_3 + f_3 + b_3) + \dot{f}_3 + \dot{b}_3$$

"system equation"

$$\ddot{x}_3 - a_{3,3}^2 x_3 - (a_{3,3}f_3 + \dot{f}_3) = a_{3,3}b_3 + \dot{b}_3$$

$$f_3(x_3) := \left[C + \frac{4}{9}\frac{\omega^2 R^5}{g} k_2(cent, elastic) \right]^{-1} \int_0^t \dot{k}_2(t-t', cent) x_3(t') dt'$$

$$\dot{f}_3(x_3) := \left[C + \frac{4}{9}\frac{\omega^2 R^5}{g} k_2(cent, elastic) \right]^{-1} \left[\int_0^t \ddot{k}_2(t-t', cent) x_3(t') dt' + \dot{k}_2(0, cent) x_3(t) \right]$$

$$\dot{f}_3(x_3) := \left[C + \frac{4}{9}\frac{\omega^2 R^5}{g} k_2(cent, elastic) \right]^{-1} \dot{k}_2(0, cent) x_3(t) +$$

$$+ \left[C + \frac{4}{9}\frac{\omega^2 R^5}{g} k_2(cent, elastic) \right]^{-1} \int_0^t \ddot{k}_2(t-t', cent) x_3(t') dt'$$

$$\dot{f}_3 = a_{3,3}\, \dot{f}_{3,1} + \dot{f}_{3,2}$$

$$\dot{f}_{3,1} := \left[C + \frac{4}{9}\frac{\omega^2 R^5}{g} k_2(cent, elastic) \right]^{-1} \dot{k}_2(0, cent) x_3(t)$$

$$\dot{f}_{3,2} := \left[C + \frac{4}{9}\frac{\omega^2 R^5}{g} k_2(cent, elastic) \right]^{-1} \int_0^t \ddot{k}_2(t-t', cent) x_3(t') dt'$$

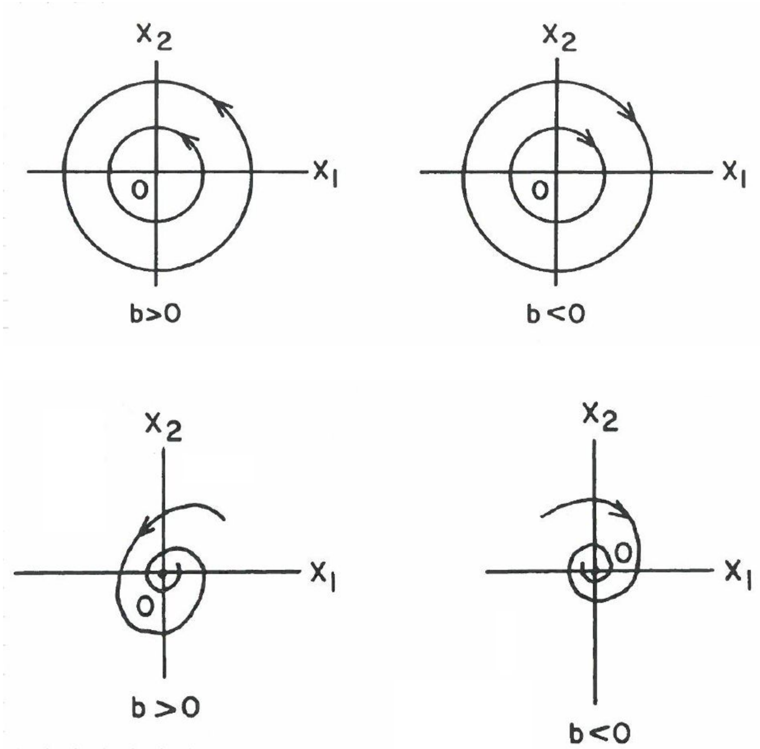
"system equation: intergro-differential equations of second order"

$$\ddot{x}_3 = (\dot{f}_{3,1} - a_{3,3}^2) x_3 - (a_{3,3}f_3 + \dot{f}_{3,2}) = a_{3,3}b_3 + \dot{b}_3$$

End of Table 2.7: Length of Day variation, second order differential

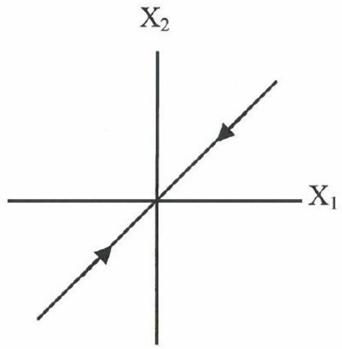
At this end, we intend to illustrate the solutions of the differential equations of second order for *Polar motion* (POM) and *Length of Day* (LOD). They generate

Figure 2.3: POM



End of Figure 2.3: POM

Figure 2.4: Excited damped, non-periodic motion of LOD



End of Figure 2.4: LOD

2.4 Laplace - and Fourier transformed incremental angular momentum balance

The *balance equations of angular momentum* are new transformed by two different kinds to the phase space. Therefore, we apply first the *Laplace transformed*, second the *Fourier transformed*. While the *Laplace transform* predicts starting form an initial time epoch $t=0$, the time behavior at another time epoch $t = t_o$. In contrast, the *Fourier transformation* applies from a time epoch $-\infty$ to a time epoch $+\infty$. This different behavior assumes that the initial state at $t=0$ in the concept of *Laplace transformation* has to be known. Therefore we have to guess the initial push or jump in the rotational motion in applying the *Laplace transformation*. When we apply the *Fourier transform* the information is not necessary, we do not need information about the history of the rotational motion, but we have live with reliable information of our model. We review therefore by *Table* 2.8 the Laplace transform and *Table* 2.4 the Fourier transform for the incremental angular momentum

Table 2.8: Laplace transformed incremental angular momentum

$$\begin{bmatrix} As\delta\breve{x}_1 + s\omega\delta\breve{j}_{1,3} + \omega(C-B)\delta\breve{x}_2 + \omega^2\delta\breve{j}_{2,3} + s\delta\breve{L}_1 - \omega\delta\breve{L}_2 \\ Bs\delta\breve{x}_2 + s\omega\delta\breve{j}_{2,3} + \omega(A-C)\delta\breve{x}_1 + \omega^2\delta\breve{j}_{1,3} + s\delta\breve{L}_2 - \omega\delta\breve{L}_1 \\ Cs\delta\breve{x}_3 + s\omega\delta\breve{j}_{3,3} + s\delta\breve{L}_3 \end{bmatrix} = \begin{bmatrix} \breve{x}_1 \\ \breve{x}_2 \\ \breve{x}_3 \end{bmatrix}$$

$$\begin{aligned} \breve{x}_1 = &-s\omega\sqrt{\frac{5}{3}}\frac{R^3}{9}(1+\breve{k}_2(load))\delta\breve{u}_{2,1}(load) - s\omega\sqrt{\frac{5}{3}}\frac{R^3}{9}\breve{k}_2(tid)\delta\breve{u}_{2,1}(tid) + \\ &-s\omega\sqrt{\frac{5}{3}}\frac{R^3}{g}\breve{k}_2(str)\delta\breve{u}_{2,1}(str) + \frac{s\omega^2R^5}{3g}\breve{k}_2(cent)\breve{y}_1 + As\breve{x}_1 + \\ &+\omega(C-B)\breve{x}_2 + \omega^2\sqrt{\frac{5}{3}}\frac{R^3}{g}(1+\breve{k}_2(load))\delta\breve{u}_{2,-1}(load) + \\ &+\omega^2\sqrt{\frac{5}{3}}\frac{R^3}{g}\breve{k}_2(tid)\delta\breve{u}_{2,-1}(tid) + \omega^2\sqrt{\frac{5}{3}}\frac{R^3}{g}\breve{k}_2(str)\delta\breve{u}_{2,-1}(str) + \\ &-\frac{\omega^3R^5}{3g}\breve{k}_2(cent)\breve{x}_2 + s\delta\breve{L}_1 - \omega\delta\breve{L}_2 \end{aligned}$$

$$\begin{aligned} \breve{x}_2 = &-s\omega\sqrt{\frac{5}{3}}\frac{R^3}{g}(1+\breve{k}_2(load))\delta\breve{u}_{2,-1}(load) - s\omega\sqrt{\frac{5}{3}}\frac{R^3}{g}\breve{k}_2(tid)\delta\breve{u}_{2,-1}(tid) + \\ &-s\omega\sqrt{\frac{5}{3}}\frac{R^3}{g}\breve{k}_2(str)\delta\breve{u}_{2,-1}(str) - \frac{s\omega^2R^5}{3g}\breve{k}_2(cent)\breve{y}_2 + Bs\breve{x}_2 + \\ &+\omega(A-C)\breve{x}_1 - \omega^2\sqrt{\frac{5}{3}}\frac{R^3}{g}(1+\breve{k}_2(load))\delta\breve{u}_{2,1}(load) + \end{aligned}$$

$$- \omega^2 \sqrt{\frac{5}{3}} \frac{R^3}{g} \breve{k}_2(tid)\delta\breve{u}_{2,-1}(tid) - \omega^2 \sqrt{\frac{5}{3}} \frac{R^3}{g} \breve{k}_2(str)\delta\breve{u}_{2,-1}(str) +$$

$$+ \frac{\omega^3 R^5}{3g} \breve{k}_2(cent)\breve{x}_1 + s\delta\breve{L}_2 - \omega\delta\breve{L}_1$$

$$\breve{x}_3 = -s\omega \frac{2\sqrt{5}}{3} \frac{R^3}{g}(1 + \breve{k}_2(load))\delta\breve{u}_{2,0}(load) + s\omega \frac{2\sqrt{5}}{3} \frac{R^3}{g} \breve{k}_2(tid)\delta\breve{u}_{2,0}(tid) +$$

$$s\omega \frac{2\sqrt{5}}{3} \frac{R^3}{g} \breve{k}_2(str)\delta\breve{u}_{2,0}(str) - s\omega^2 \frac{4}{9} \frac{R^5}{g} \breve{k}_2(cent)\breve{y}_3 + Cs\breve{x}_3 + s\delta\breve{L}_3$$

End of Table 2.8: Laplace transformed incremental angular momentum

Table 2.9: Fourier transformed incremental angular momentum

$$\begin{bmatrix} A(i\omega)\tilde{x}_1 + (i\omega)\omega_3\delta\tilde{j}_{1,3} + \omega_3(C-B)\tilde{x}_2 + \omega_3^2\delta\tilde{j}_{2,3} + (i\omega)\delta\tilde{L}_1 - \omega_3\delta\tilde{L}_2 \\ B(i\omega)\tilde{x}_2 + (i\omega)\omega_3\delta\tilde{j}_{2,3} + \omega_3(A-C)\tilde{x}_1 + \omega_3^2\delta\tilde{j}_{1,3} + (i\omega)\delta\tilde{L}_2 - \omega_3\delta\tilde{L}_1 \\ C(i\omega)\tilde{x}_3 + (i\omega)\omega_3\delta\tilde{j}_{3,3} + (i\omega)\delta\tilde{L}_3 \end{bmatrix} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix}$$

$$\tilde{x}_1 = -(i\omega)\omega\sqrt{\frac{5}{3}} \frac{R^3}{g}(1 + \tilde{k}_2(load))\delta\tilde{u}_{2,1}(load) - (i\omega)\omega\sqrt{\frac{5}{3}} \frac{R^3}{g} \tilde{k}_2(tid)\delta\tilde{u}_{2,1}(tid) +$$

$$-(i\omega)\omega\sqrt{\frac{5}{3}} \frac{R^3}{g} \tilde{k}_2(str)\delta\tilde{u}_{2,1}(str) + \frac{(i\omega)\omega^2 R^5}{3g} \tilde{k}_2(cent)\tilde{y}_1 + A(i\omega)\tilde{x}_1 +$$

$$+ \omega(C - B)\tilde{x}_2 + \omega^2 \sqrt{\frac{5}{3}} \frac{R^3}{g}(1 + \tilde{k}_2(load))\delta\tilde{u}_{2,-1}(load) +$$

$$+ \omega^2 \sqrt{\frac{5}{3}} \frac{R^3}{g} \tilde{k}_2(tid)\delta\tilde{u}_{2,-1}(tid) + \omega^2 \sqrt{\frac{5}{3}} \frac{R^3}{g} \tilde{k}_2(str)\delta\tilde{u}_{2,-1}(str) +$$

$$- \frac{\omega^3 R^5}{3g} \tilde{k}_2(cent)\tilde{x}_2 + s\delta\tilde{L}_1 - \omega\delta\tilde{L}_2$$

$$\tilde{x}_2 = -(i\omega)\omega\sqrt{\frac{5}{3}} \frac{R^3}{g}(1 + \tilde{k}_2(load))\delta\tilde{u}_{2,-1}(load) - (i\omega)\omega\sqrt{\frac{5}{3}} \frac{R^3}{g} \tilde{k}_2(tid)\delta\tilde{u}_{2,-1}(tid) +$$

$$-(i\omega)\omega\sqrt{\frac{5}{3}} \frac{R^3}{g} \tilde{k}_2(str)\delta\tilde{u}_{2,-1}(str) - \frac{(i\omega)\omega^2 R^5}{3g} \tilde{k}_2(cent)\tilde{y}_2 + B(i\omega)\tilde{x}_2 +$$

$$+ \omega(A - C)\tilde{x}_1 - \omega^2 \sqrt{\frac{5}{3}} \frac{R^3}{g}(1 + \tilde{k}_2(load))\delta\tilde{u}_{2,1}(load) +$$

$$- \omega^2 \sqrt{\frac{5}{3}} \frac{R^3}{g} \tilde{k}_2(tid)\delta\tilde{u}_{2,-1}(tid) - \omega^2 \sqrt{\frac{5}{3}} \frac{R^3}{g} \tilde{k}_2(str)\delta\tilde{u}_{2,-1}(str) +$$

$$+\frac{\omega^3 R^5}{3g}\tilde{k}_2(cent)\tilde{x}_1 + s\delta\tilde{L}_2 - \omega\delta\tilde{L}_1$$

$$\tilde{x}_3 = -(i\omega)\omega\frac{2\sqrt{5}}{3}\frac{R^3}{g}(1+\tilde{k}_2(load))\delta\tilde{u}_{2,0}(load) + (i\omega)\omega\frac{2\sqrt{5}}{3}\frac{R^3}{g}\tilde{k}_2(tid)\delta\tilde{u}_{2,0}(tid) +$$
$$+ (i\omega)\omega\frac{2\sqrt{5}}{3}\frac{R^3}{g}\tilde{k}_2(str)\delta\tilde{u}_{2,0}(str) - (i\omega)\omega^2\frac{4}{9}\frac{R^5}{g}\tilde{k}_2(cent)\tilde{y}_3 + C(i\omega)\tilde{x}_3 + (i\omega)\delta\tilde{L}_3$$

End of Table 2.9: Fourier transformed incremental angular momentum

By means of Table 2.8: Laplace transform and Table 2.9: Fourier transform we characterize in inversion process of the incremental angular momentum equations:

Table 2.10: Laplace Transformed incremental angular momentum

1st: Polar Motion

"solution to the inhomogeneous equation"

$$\begin{bmatrix} s(A+\frac{\omega^2R^5}{3g}\breve{k}_2(cent)) & \omega(C-B-\frac{\omega^2R^5}{3g}\breve{k}_2(cent)) \\ \omega(A-C+\frac{\omega^2R^5}{3g}\breve{k}_2(cent)) & s(B+\frac{\omega^2R^5}{3g}\breve{k}_2(cent)) \end{bmatrix}\begin{bmatrix}\breve{x}_1\\ \breve{x}_2\end{bmatrix}$$

$$=\begin{bmatrix} \breve{m}_1 - s\delta\breve{L}_1 - \omega\delta\breve{L}_2 + \\ +\omega\sqrt{\frac{5}{3}}\frac{R^3}{3g}\left[(s(1+\breve{k}_2(load))\delta\breve{v}_{2,1}(load) + s\breve{k}_2(tid)\delta\breve{v}_{2,1}(tid) + s\breve{k}_2(str)\delta\breve{v}_{2,1}(str) - \omega(1+\breve{k}_2(load))\delta\breve{v}_{2,-1}(load) - \omega\breve{k}_2(tid)\delta\breve{v}_{2,-1}(tid) - \omega\breve{k}_2(str)\delta\breve{v}_{2,-1}(str) \right] \\ \breve{m}_2 - s\delta\breve{L}_2 - \omega\delta\breve{L}_1 + \\ +\omega\sqrt{\frac{5}{3}}\frac{R^3}{3g}\left[(s(1+\breve{k}_2(load))\delta\breve{v}_{2,-1}(load) + s\breve{k}_2(tid)\delta\breve{v}_{2,-1}(tid) + s\breve{k}_2(str)\delta\breve{v}_{2,-1}(str) - \omega(1+\breve{k}_2(load))\delta\breve{v}_{2,1}(load) - \omega\breve{k}_2(tid)\delta\breve{v}_{2,1}(tid) - \omega\breve{k}_2(str)\delta\breve{v}_{2,1}(str) \right] \end{bmatrix}$$

$$\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\end{bmatrix} = \frac{1}{s^2(A+\frac{\omega^2R^5}{3g}\breve{k}_2(cent))(B+\frac{\omega^2R^5}{3g}\breve{k}_2(cent)) - \omega^2(A-C+\frac{\omega^2R^5}{3g}\breve{k}_2(cent))(C-B-\frac{\omega^2R^5}{3g}\breve{k}_2(cent))}\begin{bmatrix} s(B+\frac{\omega^2R^5}{3g}\breve{k}_2(cent)) & -\omega(C-B-\frac{\omega^2R^5}{3g}\breve{k}_2(cent)) \\ -\omega(A-C+\frac{\omega^2R^5}{3g}\breve{k}_2(cent)) & s(A+\frac{\omega^2R^5}{3g}\breve{k}_2(cent)) \end{bmatrix}\times$$

$$\times \left\{ \begin{bmatrix} \breve{m}_1 - s\delta\breve{L}_1 + \omega\delta\breve{L}_2 \\ \breve{m}_2 - s\delta\breve{L}_2 - \omega\delta\breve{L}_1 \end{bmatrix} + \omega\sqrt{\frac{5}{3}}\frac{R^3}{g} \begin{bmatrix} s(1+\breve{k}_2(load)) & s\breve{k}_2(tid) & s\breve{k}_2(str) & -\omega(1+\breve{k}_2(cent)) & -\omega\breve{k}_2(tid) & -\omega\breve{k}_2(str) \\ \omega(1+\breve{k}_2(load)) & \omega\breve{k}_2(tid) & \omega\breve{k}_2(str) & s(1+\breve{k}_2(cent)) & s\breve{k}_2(tid) & s\breve{k}_2(str) \end{bmatrix} \begin{bmatrix} \delta\breve{v}_{2,1}(load) \\ \delta\breve{v}_{2,1}(tid) \\ \delta\breve{v}_{2,1}(str) \\ \delta\breve{v}_{2,-1}(load) \\ \delta\breve{v}_{2,-1}(tid \\ \delta\breve{v}_{2,-1}(str) \end{bmatrix} \right\}$$

2nd: length-of-day variation

"solution to the inhomogeneous equation"

$$\tilde{x}_3 = \frac{\delta\breve{m}_3 + \omega\frac{2\sqrt{5}}{3}\frac{R^3}{g}\left[-s(1+\breve{k}_2(load))\delta\tilde{v}_{2,0}(load) - s\breve{k}_2(tid)\delta\tilde{v}_{2,0}(tid) - s\breve{k}_2(str)\delta\tilde{v}_{2,0}(str) \right] - \delta\breve{L}_3}{s(C - \frac{4\omega^2 R^5}{9g}\breve{k}_2(cent))}$$

End of Table 2.10: Laplace Transformed incremental angular momentum

Table 2.11: Fourier Transformed incremental angular momentum

1st: Polar Motion

"solution to the inhomogeneous equation"

$$
\begin{bmatrix} (i\omega)(A+\frac{\omega^2 R^5}{3g}\tilde{k}_2(cent)) & \omega(C-B-\frac{\omega^2 R^5}{3g}\tilde{k}_2(cent)) \\ \omega(A-C+\frac{\omega^2 R^5}{3g}\tilde{k}_2(cent)) & (i\omega)(B+\frac{\omega^2 R^5}{3g}\tilde{k}_2(cent)) \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}
$$

$$
= \begin{bmatrix} \tilde{m}_1 - s\delta\tilde{L}_1 - \omega\delta\tilde{L}_2 + \\ +\omega\sqrt{\frac{5}{3}}\frac{R^3}{3g}\Big[(i\omega)(1+\tilde{k}_2(load))\delta\tilde{v}_{2,1}(load) + (i\omega)\tilde{k}_2(tid)\delta\tilde{v}_{2,1}(tid) + (i\omega)\tilde{k}_2(str)\delta\tilde{v}_{2,1}(str) - \omega(1+\tilde{k}_2(load))\delta\tilde{v}_{2,-1}(load) - \omega\tilde{k}_2(tid)\delta\tilde{v}_{2,-1}(tid) - \omega\tilde{k}_2(str)\delta\tilde{v}_{2,-1}(str) \Big] \\ \tilde{m}_2 - s\delta\tilde{L}_2 - \omega\delta\tilde{L}_1 + \\ +\omega\sqrt{\frac{5}{3}}\frac{R^3}{3g}\Big[(i\omega)(1+\tilde{k}_2(load))\delta\tilde{v}_{2,-1}(load) + (i\omega)\tilde{k}_2(tid)\delta\tilde{v}_{2,-1}(tid) + (i\omega)\tilde{k}_2(str)\delta\tilde{v}_{2,-1}(str) - \omega(1+\tilde{k}_2(load))\delta\tilde{v}_{2,1}(load) - \omega\tilde{k}_2(tid)\delta\tilde{v}_{2,1}(tid) - \omega\tilde{k}_2(str)\delta\tilde{v}_{2,1}(str) \Big] \end{bmatrix}
$$

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \frac{2}{(i\omega)^2 (A + \frac{\omega^2 R^5}{3g}\tilde{k}_2(cent))(B + \frac{\omega^2 R^5}{3g}\tilde{k}_2(cent)) - \omega^2 (A - C + \frac{\omega^2 R^5}{3g}\tilde{k}_2(cent))(C - B - \frac{\omega^2 R^5}{3g}\tilde{k}_2(cent))} \begin{bmatrix} (i\omega)(B + \frac{\omega^2 R^5}{3g}\tilde{k}_2(cent)) & -\omega(C - B - \frac{\omega^2 R^5}{3g}\tilde{k}_2(cent)) \\ -\omega(A - C + \frac{\omega^2 R^5}{3g}\tilde{k}_2(cent)) & (i\omega)(A + \frac{\omega^2 R^5}{3g}\tilde{k}_2(cent)) \end{bmatrix}$$

$$\times \left\{ \begin{bmatrix} \tilde{m}_1 - (i\omega)\delta\tilde{L}_1 + \omega\delta\tilde{L}_2 \\ \tilde{m}_2 - (i\omega)\delta\tilde{L}_2 - \omega\delta\tilde{L}_1 \end{bmatrix} + \omega\sqrt{\frac{5}{3}}\frac{R^3}{g} \begin{bmatrix} (i\omega)(1+\tilde{k}_2(load)) & (i\omega)\tilde{k}_2(tid) & (i\omega)\tilde{k}_2(str) & -\omega(1+\tilde{k}_2(cent)) & -\omega\tilde{k}_2(tid) & -\omega\tilde{k}_2(str) \\ \omega(1+\tilde{k}_2(load)) & \omega\tilde{k}_2(tid) & \omega\tilde{k}_2(str) & (i\omega)(1+\tilde{k}_2(cent)) & (i\omega)\tilde{k}_2(tid) & (i\omega)\tilde{k}_2(str) \end{bmatrix} \begin{bmatrix} \delta\tilde{v}_{2,1}(load) \\ \delta\tilde{v}_{2,1}(tid) \\ \delta\tilde{v}_{2,1}(str) \\ \delta\tilde{v}_{2,-1}(load) \\ \delta\tilde{v}_{2,-1}(tid \\ \delta\tilde{v}_{2,-1}(str) \end{bmatrix} \right\}$$

2nd: length-of-day variation

"solution to the inhomogeneous equation"

$$\tilde{x}_3 = \frac{\delta\tilde{m}_3 + \omega\frac{2\sqrt{5}}{3}\frac{R^3}{g}\left[-(i\omega)(1+\tilde{k}_2(load))\delta\tilde{v}_{2,0}(load) - (i\omega)\tilde{k}_2(tid)\delta\tilde{v}_{2,0}(tid) - (i\omega)\tilde{k}_2(str)\delta\tilde{v}_{2,0}(str) \right] - \delta\tilde{L}_3}{(i\omega)(C - \frac{4\omega^2 R^5}{9g}\tilde{k}_2(cent))}$$

End of Table 2.11: Fourier Transformed incremental angular momentum

Before we study in detail the *Chander wobble* in the next section, we determine the *Zero determinant* of the two *polar motion* components for the *resonance oscillation.*

Table 2.12: Chander wobble resonance

$$S = \frac{3g(A-C) + k_2(cent)\omega^2 R^5}{3Ag + k_2(cent)\omega^2 R^5}$$

End of Table 2.12: Chander wobble resonance

The backward transformation is illustrated here only for the load potential excitation due to *J. Engels* (2006). A typical example for the analysis of *Polar Motion* is finally given by *H. Schuh*, *S. Nagel* and *T. Seitz* (2001).

Table 2.13: Laplace forward and backward transformation (*J. Engels* 1998) for Polar Motion

"forward transformation"

$$\tilde{m} := \frac{\sqrt{30}}{R_E^2}(A_o + \sum_{j=1} \frac{A_j}{s-a_j} C_{2,-1}^{\sim V1(load)(0)} R_E^2$$

"backward transformation"

$$m(t) := \frac{\sqrt{30}}{R_E^2 \Omega^2} A_o R_E^2 C_{2,-1}^{\sim V1(load)}(t) + R_E^2 \sum_{j=1} A_j \int_0^t \exp[a_j(t-t')] C_{2,-1}^{\sim V1(load)(0)}(t')\, dt'$$

Special Case: Heaviside load:

$$R_E^2 C_{2,-1}^{\sim V1(load)(0)} = CH(t-t_o)$$

$$m(t) := -\frac{\sqrt{30}}{R_E^2 \Omega^2} H(t-t_o) \left[A_o + \sum_{j=1} A_j \exp(a_j t) I - \frac{1}{a_j} \exp(-a_j t') \right]_{t=t_o}^{t'=t} =$$

$$= -\frac{\sqrt{30}}{R_E^2 \Omega^2} H(t-t_o) \left[A_o + \sum_{j=1} A_j \frac{\exp(-a_j t_o) I - \exp(-a_j t)}{a_j} \right] =$$

$$= -\frac{\sqrt{30}}{R_E^2 \Omega^2} H(t-t_o) \left[A_o + \sum_{j=1} \frac{A_j}{a_j} (a_j(t-t_o) - 1 \right]$$

End of Table 2.13: Laplace forward and backward transformation

Table 2.14: Laplace backward transformation — into (m , m_v) for Polar Motion (*J. Engels* 1998)

$$m_x + img = -\frac{\sqrt{30}}{R_E^2 \Omega^2} H(t-t_o)\Big[\frac{A_o}{\sqrt{2}}(V_{2,1}^{Load} + iV_{2,-1}^{Load}) +$$

$$+\sum_{j=1} A_j \int_0^t \exp(a_{j,I}(t-t')(\cos(a_{j,R}(t-t') + i\sin(a_{j,R}(t-t'))) \frac{1}{\sqrt{2}} V_{2,1}^{Load}(t')dt' \Big] =$$

$$= -\frac{\sqrt{15}}{R_E^2 \Omega^2}\Big[A_{o,R}V_{2,1}^{Load} - A_{o,1}V_{2,-1}^{Load} + i(A_{o,R}V_{2,-1}^{Load} + A_{o,I}V_{2,-1}^{Load}) +$$

$$+\sum_{j=1}\int_0^t \exp(a_{j,R}(t-t')\Big\{ \cos(a_{j,I}(t-t')V_{2,1}^{Load}(t') - \sin(a_{j,I}(t-t')V_{2,-1}^{Load}(t') +$$

$$+ i\cos(a_{j,I}(t-t')V_{2,1}^{Load}(t') - \sin(a_{j,I}(t-t')V_{2,-1}^{Load}(t')\Big\}(A_{j,R} + A_{j,I})dt' \Big] \Rightarrow$$

$$(i) \qquad m_x = -\frac{\sqrt{15}}{R_E^2 \Omega^2}\Big[A_{o,R}V_{2,1}^{Load} - A_{o,I}V_{2,-1}^{Load} +$$

$$+\sum_{j=1}\int_0^t \exp(a_{j,R}(t-t')\Big\{ \cos(a_{j,I}(t-t')(A_{j,R}V_{2,1}^{Load}(t') - A_{j,I}V_{2,-1}^{Load}(t')) +$$

$$+ i\sin(a_{j,I}(t-t')(-A_{j,R}V_{2,1}^{Load}(t') - A_{j,I}V_{2,-1}^{Load}(t'))\Big\} dt'$$

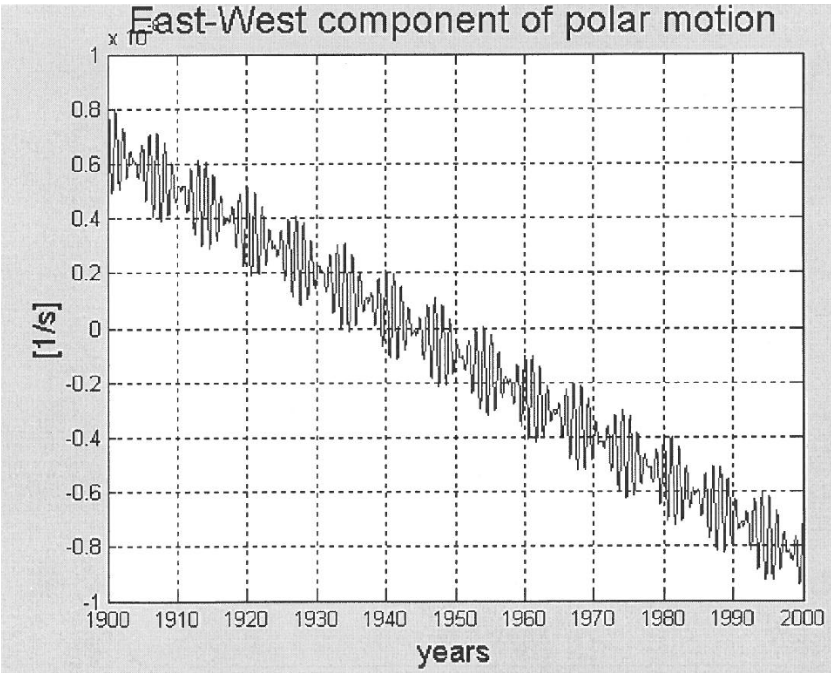
$$(ii) \qquad m_y = -\frac{\sqrt{15}}{R_E^2 \Omega^2}\Big[A_{o,R}V_{2,1}^{Load}(t) + A_{o,I}V_{2,-1}^{Load}(t) +$$

$$+\sum_{j=1}\int_0^t \exp(a_{j,R}(t-t')\Big\{ \cos(a_{j,I}(t-t')(A_{j,R}V_{2,-1}^{Load}(t') + A_{j,I}V_{2,1}^{Load}(t')) +$$

$$+ i\sin(a_{j,I}(t-t')(A_{j,R}V_{2,1}^{Load}(t') - A_{j,I}V_{2,-1}^{Load}(t'))\Big\} dt'$$

End of Table 2.14: Linear drift and periodic variations observed in any time series of polar motion, H. Schuh, S. Nagel and T. Seitz (2001) J of Geodesy 74 (2001) 701-710

Figure 2.5: Linear drift and periodic variations observed in any time series of polar motion, H. Schuh, S. Nagel and T. Seitz (2001) J of Geodesy 74 (2001) 701-710



3. Summary

At first, we heigh lighted the basic work of A. Dermanis with respect to Geodetic Reference Frames starting with his Ph.D. Thesis on VLBI. His deformation analysis paved the way for characteristics like dilatation, shear, rotation and energy of central importance, namely Frame Invariance and Parameter Estimability of key importance for up-to-date Geodesy in his work on transformation parameters between various geodetic frames. We embed our own work, which relates to him.

Second, we present to you nearly our System Theory of polar Motion (POM) and Length of Day Variations (LOD). In case of two identical eigenvalues of the Inertia Tensor we proved:

Polar Motion is generating an excited circular harmonic oscillator.

This result has to generalized for the general case of three different eigenvalues of the second order inertia tensor:

Polar Motion is generated by an excited elliptic harmonic oscillator.

In contrast, Length of Day Variations (LOD) are best described by:

Excited damped un harmonics (non-periodic) motion.

We have a technique for analysis pioneered by *M. Schneider* (1999) who differentiated first order system equations second order differential equations. They can be more easily been solved. For the interpretation of the various system equations for POM and LOD we finished our short review with *Laplace and Fourier transformed incremental angular momentum balance*. Damped or alternatively periodic ones are perfectly described.

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