An Introduction to Tropical Convexity

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Preface: This paper is written in the frame of the seminar Advanced Topics in Discrete Mathematics held during 2004 at ETH Zurich. The layout as well as the content is mostly based on Develin and Sturmfels (2004) and on Sturmfels (2002) and Speyer and Sturmfels (2004). After an introduction to the tropical semiring and its operations, the convexity notion will be redefined in this tropical setting. Following that, tropical polytopes will be introduced together with some of their properties. Tropical cell complexes will be discussed in the sequel. The paper will conclude with an application area of tropical mathematics in control of discrete event systems. Throughout the paper, the emphasis is on the geometrical intuition. Proofs will be given only in cases of special interest.

Key words: tropical mathematics, polytopes, convexity, max-plus algebra, control.

1. Introduction

The *tropical semiring* $(R, \oplus, \odot)$ is the set of real numbers with the operations of tropical addition, which takes the minimum of two numbers and tropical multiplication, which takes their sum. In short:

$$a \oplus b := \mathit{min}(a, b) \quad \text{and} \quad a \odot b := a + b.$$

The above generalize in R^n as pointwise minimum and (scalar) pointwise addition operations, with tropical addition

$$(x_1, \ldots, x_n) \oplus (y_1, \ldots, y_n) = (x_1 \oplus y_1, \ldots, x_n \oplus y_n),$$

and tropical multiplication with a scalar c

$$c \odot (x_1, x_2, \ldots, x_n) = (c \odot x_1, c \odot x_2, \ldots, c \odot x_n).$$

Sometimes instead of the minimum, the maximum is considered in the definition of tropical addition. A set S of R^n is called *tropically convex* if

$$\forall a, b \in R, \forall x, y \in S: \; a \odot x \oplus b \odot y \in S.$$

The *tropical convex hull* of a given subset $V \subset R^n$ is the smallest tropically convex subset of R^n which contains V. Any tropically convex subset S of R^n is closed under the tropical scalar multiplication. That is, if a point $x \in S$, then also $x + \lambda (1, \ldots, 1) \in S$, for all $\lambda \in R$. This property gives rise to the idea of *tropical projective space* TP^{n-1} which is defined TR^{n-1} as

$$TP^{n-1} = R^n / (1, \ldots, 1) R.$$

In other words, we present any tropically convex set in R^n by its projection on TP^{n-1}. We can think about this projection as resulting from a translation along the diagonal $(1, 1, \ldots, 1) \in R^n$. In particular, we can normalize a point by translation until its first coordinate becomes zero.

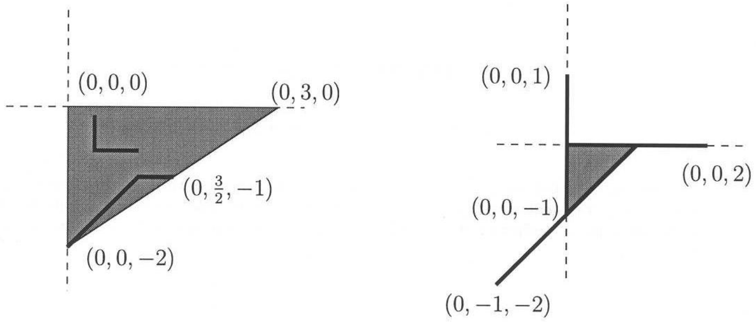

Figure 1. *Tropical convex set and tropical polytope*

For example, the point $(x_1, x_2, x_3, x_4) \in R^4$ will be represented in TP^3 by the coordinates $(0, x_2 - x_1, x_3 - x_1, x_4 - x_1)$, i.e. by $(x_2 - x_1, x_3 - x_1, x_4 - x_1)$. Also note, that while the translation must be performed along the diagonal, the final projection can be onto any hyperplane that properly intersects the diagonal. The former approach facilitates computation, while the latter helps visualization.

A *tropical polytope* is the tropical convex hull of a *finite* subset V (i.e. a finite set of points) in TP^{n-1}.

2. Tropically convex sets

Let's begin with two pictures of convex sets in the tropical plane TP^2. A point $(x_1, x_2, x_3) \in TP^2$ is represented by drawing the differences $(x_2 - x_1, x_3 - x_1)$. For convex sets, this is all the information needed, as a consequence of the fact that tropically convex sets are closed under tropical scalar multiplication. The left part of figure 1 is not a tropical polytope, because it is *not* the tropical convex hull of *finitely* many points. However, it is a tropically convex set. On the other hand, the right part of figure 1 is a tropical polytope. It is the tropical convex hull of the three points, whose coordinates are also shown. The thick black lines in the left subfigure represent tropical line segments. To get a better intuition for these geometrical objects, we first have to see what a tropical line and a tropical line segment really are. We begin with some definitions.

Definition 2.1. A tropical polynomial f (x) is the minimum of a finite set of linear functions with N-coefficients.

For example,

$$f(x, y) = 5 \cdot x^2 y \oplus xy \oplus 0 \cdot xy^3$$

actually means

$$f(x, y) = \min \{5 + 2x + y, x + y, 0 + x + 3y\}$$

In this tropical setting, the additive unit is $+\infty$. That is, $x \oplus (+\infty) = \min\{x, +\infty\} = x$. Therefore, the tropical line will be defined as the tropical variety of a linear polynomial, that is, the set of points satisfying the following equation:

$$f(x, y) = ax \oplus by \oplus c = +\infty \tag{2.1}$$

where a, b, c are fixed real numbers. Here, we have to trust that in ordinary arithmetic, this amounts to finding those points for which the minimum is attained

$$\min \{x + a, y + b, c\}$$

least twice. The tropical lines, due to the way they are constructed, consist of regular line segments, and rays whose slopes are zero-one vectors. This generalizes to higher dimensions, where instead of tropical lines we have tropical hyperplanes. Definition 2.1 refers to R^n space.

Now we are going to give a more precise description of how tropical line segments really look like. Conceptually, one has to take two points in R^n and construct their convex hull, i.e., for two points $x, y \in R^n$ the tropical line segment between x and y is the set $\{a \odot x \oplus b \odot y \mid a, b \in R\}$. Then, project it down to TP^{n-1} and look at the result. Taking a closer look at the conceptual procedure just described, we arrive at the following result:

Proposition 2.2. *The tropical line segment between two points* x *and* y *in* TP^{n-1} *is the concatenation of at most* $n - 1$ *ordinary line segments. The slope of each line segment is a zero-one vector.*

Proof. After relabeling the coordinates of $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ we may assume

$$y_1 - x_1 \leq y_2 - x_2 \leq \cdots \leq y_n - x_n \tag{2.2}$$

The following points lie in the given order on the tropical segment between x and y:

$$x = (y_1 - x_1) \odot x \oplus y = (y_1, y_1 - x_1 + x_2, y_1 - x_1 + x_3, \ldots, y_1 - x_1 + x_{n-1}, y_1 - x_1 + x_n)$$

$$(y_2 - x_2) \odot x \oplus y = (y_1, y_2, y_2 - x_2 + x_3, \ldots, y_2 - x_2 + x_{n-1}, y_2 - x_2 + x_n)$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$(y_{n-1} - x_{n-1}) \odot x \oplus y = (y_1, y_2, y_3, \ldots, y_{n-1}, y_{n-1} - x_{n-1} + x_n)$$

$$y = (y_n - x_n) \odot x \oplus y = (y_1, y_2, y_3, \ldots, y_{n-1}, y_n).$$

Between any two consecutive points, the tropical line segment agrees with the ordinary line segment, which has slope $(0, 0, \ldots, 0, 1, 1, \ldots, 1)$. Hence the tropical line segment between x and y is the concatenation of at most $n - 1$ ordinary line segments, one for each strict inequality in (2.2).

This description of tropical segments shows an important feature of tropical polytopes: their edges use a limited set of directions. A tropical polytope in TP^{n-1} is nothing more than the tropical convex hull of a **finite** number of points. Conceptually, the construction of the tropical convex hull of a set of points in TP^{n-1} can be based on the following proposition.

Let $v_1, \ldots, v_r \in V$ and $a_1, \ldots, a_r \in R$. Then the set of all tropically linear combinations

$$a_1 \odot v_1 \oplus a_2 \odot v_2 \oplus \ldots \oplus a_r \odot v_r \tag{2.3}$$

satisfy the following proposition.

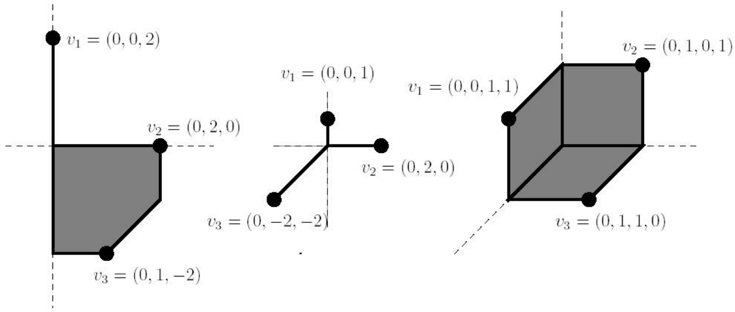

Figure 2. *Three tropical polytopes. First two in TP^2, the last in TP^3*

Proposition 2.3. *The smallest tropically convex subset of* TP^{n-1} *which contains a given set V coincides with the set of all tropical linear combinations (2.3). We denote this set by tconv(V).*

Proof. By definition, the smallest tropically convex subset of TP^{n-1} which contains a given set V, is its tropical convex hull. To show that it coincides with the set of all tropically linear combinations (2.3), we have to show two things: First, that the set of all linear combinations are contained in the convex hull and second, that the convex hull is contained in the set of all linear combinations. For the first, we argue by induction on the number of points r. Let $x = \oplus^{r}_{i=1} a_i \odot v_i$ be the point in (2.3). If $r \leq 2$, then x is clearly in the tropical convex hull of V (because the tropical convex hull is also a tropically convex set, therefore it has to contain all the points given by $a \odot x \oplus b \odot y$ for all $x, y \in V$ and $a, b \in R$ - see definition on the first page). Now, if $r > 2$, we write $x = a_1 \odot v_1 \oplus (\oplus^{r}_{i=2} a_i \odot v_i)$. The parenthesized vector lies in the tropical convex hull, by induction on r, and hence so does x. For the converse, consider any two tropical linear combinations

$x = \oplus^{r}_{i=2} c_i \odot v_i$ and $y = \oplus^{r}_{i=1} d_i \odot v_i$. By the distributive law, $a \odot x \oplus b \odot y$ is also a tropical linear combination of $v_1, v_2, \ldots, v_r \in V$. Hence the set of all tropical linear combinations is tropically convex, so it contains the tropical convex hull of V. (Here, we make use of the fact that the tropical convex hull is the smallest tropically convex set that contains a given set of points, therefore it is contained in every other convex set containing them).

If V is a finite subset of TP^{n-1} then tconv(V) is a *tropical polytope*. Figure 2 shows some examples of that. The two first are tropical convex hulls of three points each and the last one is the union of three squares. Carathéodory's theorem also holds in the tropical case.

Theorem 2.4. (Tropical Carathéodory's theorem). *If x is in the tropical convex hull of a set of r points* v_i **in** TP^{n-1}**,** *then x is in the tropical convex hull of at most n of them.*

Generalizing the notion of the tropical line, a tropical hyperplane defined by a tropical linear form $a_1 \odot x_1 \oplus a_2 \odot x_2 \oplus \cdots \oplus a_n \odot x_n$ consists of all points $x = (x_1, x_2, \ldots, x_n)$ in TP^{n-1} such that the following holds (in ordinary arithmetic)

$$a_i + x_i = a_j + x_j = \min\{a_k + x_k : k = 1, \ldots, n\} \tag{2.4}$$

for some indices $i \neq j$. That is, it consists of those points at which the minimum in (2.4) is attained more than once (at least twice). As might have been expected, the following proposition holds.

Proposition 2.5. *Tropical hyperplanes in* TP^{n-1} *are tropically convex.*

Proof. Let H be a hyperplane defined by (2.4). Suppose that two points x and y lie in H and consider any tropical linear combination $z = c \odot x \oplus d \odot y$. To show that H is tropically convex, we have to show that $z \in H$. For this to hold, if we plug the coordinates of $z = (z_1, z_2, \ldots, z_n)$ into the linear form defining H, the corresponding minimum must be attained at least twice. Let i be the index which minimizes $a_i + z_i$.

We need to show that this minimum is attained at least twice. By definition, z_i is equal to either $c + x_i$ or $d + y_i$. After permuting the coordinates of x and y, we may assume without loss of generality that $z_i = c + x_i$. Remember that i is a fixed index that minimizes the expression $a_k + z_k$ over $k = 1, \ldots, n$. By this construction, we have also $z_k \leq c + x_k$ for all $k = 1, \ldots, n$. Hence, $a_i + z_i \leq a_k + z_k \leq a_k + c + x_k$ for all $k = 1, \ldots, n$. From this it follows that $a_i + x_i \leq a_k + x_k$ for all k so that $a_i + x_i$ achieves the minimum of $\{a_1 + x_1, a_2 + x_2, \ldots, a_n + x_n\}$. But since $x \in H$, there exists a second index $j \neq i$ for which this minimum is attained twice: $a_j + x_j = a_i + x_i$. But now $a_j + z_j \leq a_j + c + x_j = c + a_i + x_i = a_i + z_i$. Since $a_i + z_i$ is the minimum of all $a_j + z_j$, the two must be equal and this minimum is attained at least twice, as desired.

Proposistion 2.5 implies that if V is a subset of TP^{n-1} which happens to lie in a tropical hyperplane H, then its tropical convex hull tconv(V) will lie in H as well. Yet another relevant object is the *tropical plane* which can be expressed as the intersection of tropical hyperplanes. But not any arbitrary intersection of any set of hyperplanes qualifies as tropical plane. This is a fine point of tropical geometry.

3. Tropical polytopes and cell complexes

Throughout this section we fix a finite subset $V = (v_1, v_2, \ldots, v_r)$ of r points in the tropical projective space TP^{n-1}. Here, $v_i = (v_{i1}, v_{i2}, \ldots, v_{in})$. We will study the tropical polytope P = tconv(V). Our goal is to describe a natural cell decomposition of TP^{n-1} induced by the fixed finite subset V.

Let x be any point in TP^{n-1}. The *type* of x relative to V is the ordered n-tuple $(S_1, S_2, \ldots, S_n)$ of subsets $S_j \subseteq \{1, 2, \ldots, r\}$ which is defined as follows: An index i is in S_j if

$$v_{ij} - x_j = \min\{v_{i1} - x_1, v_{i2} - x_2, \ldots, v_{in} - x_n\} \tag{3.1}$$

Equivalently, if we set $\lambda_i = \min \{\lambda \in R : \lambda \odot v_i \oplus x = x\}$ then S_j is the set of all indices i such that $\lambda_i \odot v_i$ and x have the same j-th coordinate. We say that a n-tuple of index sets $S = (S_1, S_2, \ldots, S_n)$ is a type if it arises in this manner. Note that from the definition of the type it follows that every i will eventually be included in some S_j.

Remark 3.1. See the notes in the Appendix for an example plus a proof of the equivalence of the two type definitions and some more intuition for the *type*.

We now state and prove the tropical Farkas Lemma.

Proposition 3.2 (Tropical Farkas Lemma). *For all* $x \in TP^{n-1}$*, exactly one of the following is true.*

(i) *the point x is in the tropical polytope P = tconv(V), or*

(ii) *there exists a tropical hyperplane which separates x from P.*

We have to clarify what the aforementioned separation statement really means. It means the following: if the hyperplane is given by (2.4) and $a_k + x_k = \min\{a_1 + x_1, \dots, a_n + x_n\}$ then $a_k + y_k > \min\{a_1 + y_1, \dots, a_n + y_n\}$ for all $y \in P$.

Proof. Consider any point $x \in TP^{n-1}$, with type type(x) = $(S_1, \dots, S_n$, and let $\lambda_i = \min\{\lambda \in R : \lambda \odot v_i \oplus x = x\}$ as before. We define

$$\pi_V(x) = \lambda_1 \odot v_1 \oplus \lambda_2 \odot v_2 \oplus \cdots \oplus \lambda_r \odot v_r . \tag{3.2}$$

There are two cases: either $\pi_V(x) = x$ or $\pi_V(x) \neq x$. The first case implies (i), because then (3.2) expresses x as a tropical linear combination of the points in V, hence x lies in tconv(V) = P. Since (i) and (ii) clearly cannot occur at the same time, it suffices to prove that the second case implies (ii).

Suppose that $\pi_V(x) \neq x$. Then following lemma holds:

Lemma 3.3. *There exists some index* $k \in \{1, \dots, n\}$ *such that* $v_{ik} + \lambda_i - x_k > 0$ *for every* $i = 1, \dots, r$.

Proof. We begin by expanding the expression $\pi_V(x) \neq x$. It denotes the following:

$$\begin{bmatrix} \min(\lambda_1 + v_{11}, \lambda_2 + v_{21}, \dots, \lambda_r + v_{r1}) \\ \min(\lambda_1 + v_{12}, \lambda_2 + v_{22}, \dots, \lambda_r + v_{r2}) \\ \vdots \\ \min(\lambda_1 + v_{1n}, \lambda_2 + v_{2n}, \dots, \lambda_r + v_{rn}) \end{bmatrix} \neq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{3.3}$$

We also know (see remark 3.1) that $\lambda_i + v_{ij} \geq x_j$ for all i, j. This fact together with (3.3) means that the equality in (3.3) cannot be achieved because for some (at least one) j, say j = k equality cannot be achieved. Hence, there exists $k \in \{1, \dots, n\}$ such that $\lambda_i + v_{ik} > x_k$ for all $i \in \{1, \dots, r\}$. This also means that S_k will be empty for the same k.

Now, we choose an $\varepsilon > 0$ such that $\varepsilon < v_{ik} + \lambda_i - x_k$ for all $i = 1, \dots, r$. We choose the separating hyperplane (2.4) as follows:

$$a_k := -x_k - \varepsilon \quad \text{and} \quad a_j := -x_j \quad \text{for } j \in \{1, \dots, n\} \backslash k. \tag{3.4}$$

This choice clearly satisfies $a_k + x_k = \min\{a_1 + x_1, \dots, a_n + x_n\}$. Now, consider any point $y = \oplus^{r}_{i=1} c_i \odot v_i$ in tconv(V). In more detail, this is equal to the following

$$y = \begin{bmatrix} \min(c_1 + v_{11},...,c_m + v_{m1},...,c_r + v_{r1}) \\ \cdots \\ \min(c_1 + v_k,...,c_m + v_{mk},...,c_r + v_{r1}) \\ \cdots \\ \min(c_1 + v_{1n},...,c_m + v_{mn},...,c_r + v_{rn}) \end{bmatrix}$$

Pick any m such that $y_k = c_m + v_{mk}$. By definition of the λ_i , we have $x_k \leq \lambda_m + u_{mk}$ for all k and there exists some j with $x_j = \lambda_m + v_m$ j. All the above, put together, imply

$$a_k + y_k = a_k + c_m + v_{mk} = c_m + v_{mk} - x_k - \varepsilon > c_m - \lambda_m$$
$$= c_m + v_{mj} - x_j \geq y_j - x_j = a_j + y_j \geq \min(a_1 + y_1, . . . , a_n + y_n$$

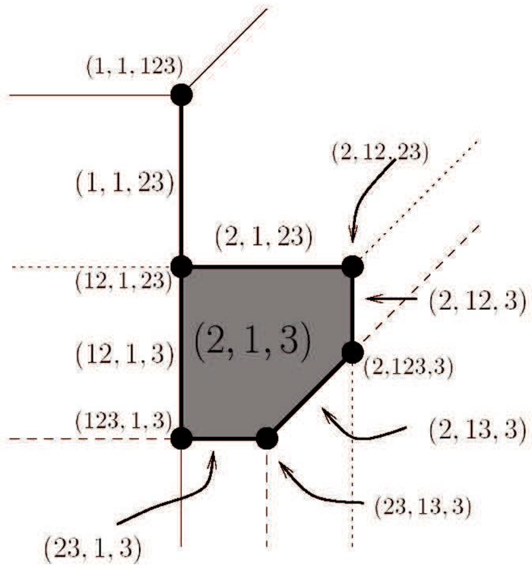

Figure 3. *Geometrical construction of the cell decomposition C_V*

Therefore, the hyperplane defined by (2.4) separates x from P as desired.

We will now sketch the idea of the *tropical cell complex*. Let $S = (S_1, . . . , S_n)$ as before and consider the set of all points whose type contain S

$$X_S := \{x \in TP^{n-1} : S \subseteq \text{type}(x)\}.$$

Lemma 3.4. *The set* X_S *is a closed convex polyhedron (in the usual sense).*

We are now prepared to state the following important theorem (no more proofs).

Theorem 3.5. *The collection of convex polyhedra* X_S, *where* S ranges over all possible types, defines a cell decomposition C_V *of* TP_{n-1}. *The tropical polytope*

$P = \mathrm{tconv}(V)$ *equals the union of all bounded cells* X_S *in this decomposition.*

This is clarified by figure 3. There is also a nice geometrical construction of this decomposition, briefly given by the following proposition.

Proposition 3.6. *The cell decomposition* C_V *is the common refinement of the* r *fans* $-F + v_i$.

F in the proposition above is the fan defined by the tropical hyperplane (2.4). Using the idea of the cell decomposition, the following proposition can be proven.

Proposition 3.7. *If* P *and* Q *are tropical polytopes in* TP^{n-1} *then* $P \cap Q$ *is also a tropical polytope.*

This concludes the section.

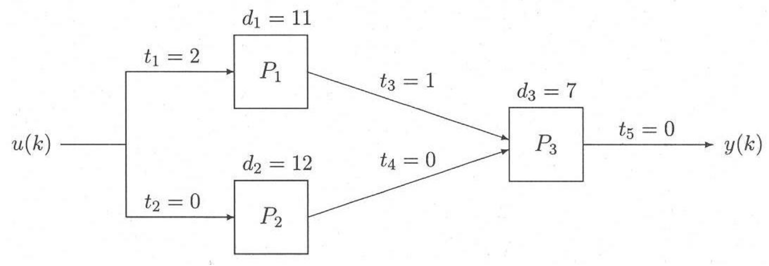

Figure 4. *A simple manufacturing system*

4. Max plus algebra and control

In this section, we'll try to briefly present the connection between tropical mathematics, also known as max-plus or min-plus algebra, with control engineering. Some of the references on the subject include, but are not limited to, Cohen et al. (1999) and De Schutter and van den Boom (2000). More specifically, the max-plus algebra framework can be used to model discrete-event systems. The main advantage of this modelling technique is that the normally viewed nonlinear dynamics of the system become linear in the tropical setting. For example, consider the production system of figure (4).

This manufacturing system consists of three processing units: P_1, P_2 and P_3 and works in batches (one batch for each finished product). Raw material is fed to P_1 and P_2, processed and sent to P_3 where assembly takes place. Note that each input batch of raw material is split into two parts: one part of the batch goes to P_1 and the other part goes to P_2. The processing times for P_1, P_2 and P_3 are respectively $d_1 = 11$, $d_2 = 12$ and $d_3 = 7$ time units. We assume that it takes $t_1 = 2$ time units for the raw material to get from the input source to P_1, and $t_3 = 1$ time unit for a finished product of P_1 to get to P_3. The other transportation times (t_2, t_4 and t_5) and the set-up times are assumed to be negligible. At the input of the system and between the processing units there are buffers with a capacity that is large enough to ensure that no buffer overflow will occur. A processing unit can only start working on a new product if it has finished processing the previous product. We assume that each processing unit starts working as soon as all parts are available.

Now we want to derive a model for this system, related to tropical mathematics. It has been shown that discrete event systems with no concurrency and only synchronization can be modelled by a max-plus algebraic model of the following form:

$$\begin{aligned} x(k+1) &= A \odot x(k) \oplus B \odot u(k) \\ y(k) &= C \odot x(k) \end{aligned} \tag{4.1}$$

For a manufacturing system, u(k) would typically represent the time instant at which the raw material is fed to the system for the $(k+1)^{th}$ time, x(k) the time instants at which the machines start processing the k^{th} batch of intermediate products, and y(k) the time instants at which the k^{th} batch of finished products leaves the system. A discrete-event system that can be modelled by (4.1) is sometimes called a *max-plus-linear* time-invariant discrete-event system, or MPL system for short.

Now we derive the max-plus-linear state space model of the production system of figure (4). First we determine the time instant at which processing unit P_1 starts working for the $(k+1)^{th}$ time. If we feed raw material to the system for the $(k+1)^{th}$ time, then this raw material is available at the input of processing unit P_1 at time $t = u(k) + 2$. However, P_1 can only start working on the new batch of raw material as soon as it has finished processing the current, i.e. the k^{th} batch. Since the processing time on P_1 is $d_1 = 11$ time units, the k^{th} intermediate product will leave P_1 at time $t = x_1(k) + 11$. Since P_1 starts working on a batch of raw material as soon as the raw material is available and the current batch has left the processing unit, this implies that we have

$$\underline{x_1}(k+1) = \max(x_1(k) + 11, u(k) + 2). \tag{4.2}$$

Using a similar reasoning we find the following expressions for the time instants at which P_2 and P_3 start working for the $(k+1)^{st}$ time and for the time instant at which the k^{th} finished product leaves the system:

$$\begin{aligned} x_2(k+1) &= \max(x_2(k) + 12, u(k) + 0) \\ x_3(k+1) &= \max(x_1(k+1) + 11 + 1, x_2(k+1) + 12 + 0, \ x_3(k) + 7) \\ &= \max(x_1(k) + 23, x_2(k) + 24, x_3(k) + 7), u(k) + 14) \\ y(k) &= x_3(k) + 7 + 0. \end{aligned} \tag{4.3}$$

By using $-\infty$ as the zero element of tropical addition (note that we use a max convention in this example), we can rewrite the preceding equations in a matrix format resembling linear time invariant systems.

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} 11 & -\infty & -\infty \\ -\infty & 12 & -\infty \\ 23 & 24 & 7 \end{bmatrix} \odot x(k) \odot \begin{bmatrix} 2 \\ 0 \\ 14 \end{bmatrix} \odot u(k) \\
y(k) &= \begin{bmatrix} -\infty & -\infty & 7 \end{bmatrix} \odot x(k)
\end{aligned}
\tag{4.4}
$$

These models are quite appealing, because they have a nice"linear" structure which is amenable to computations. Starting from these models, various control schemes can be applied, including model predictive control (De Schutter and van den Boom, 2000).

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Appendix

A. Clarifications concerning the 'type' definition.

i: refers to the n-vectors v_i of TP^{n-1}, i = 1, ….. r

j: refers to the j^{th} coordinate of x or v_i, j = 1, ….. r

Index set S_j → refers to some coordinate, e.g. 4^{th}, 6^{th}.

Example: let n=r=3 and

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \quad x = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(i=1) (i=2) (i=3)

a) We take the differences, $v_i - x$, and b) ask where is the minimum at? what j? i.e.,

$$v_1 - x = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \leftarrow j^* = 2, \quad v_2 - x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{matrix} \leftarrow j^* = 1 \\ \\ \\ \end{matrix}, \quad v_3 - x = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow j^* = 3 \end{matrix}$$

c) For each coordinate j=1,…n, we ask the following questions,

S_1: j=1. For which i's does j equal j*?
For i=2 only. Therefore $S_1 = \{2\}$

S_2: j=2. For which i does j* =2?
For i=1 only. Hence $S_2 = \{1\}$

S_3: j=3. In similar manner, $S_3 = \{3\}$

In total we have $S = \{\{2\}, \{1\}, \{3)\}$.

B. Equivalence of the second definition

To every vector v_i from the set V, and with respect to an arbitrary fixed point x, we attach a scalar λ_i, defined as follows:

$$\lambda_i := \min \{ \lambda \in R : \lambda \odot v_i \oplus x = x \}$$

Let's elaborate on it:

$$\lambda \odot v_i = \lambda \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} v_{i1} \\ \vdots \\ v_{in} \end{bmatrix}, \quad \text{so "}\lambda \odot v_i \oplus x = x\text{" means that elementwise,}$$

coordinatewise, for all coordinates, the coordinate values of x are less or equal those of $\lambda \odot v_i$. To put it short:

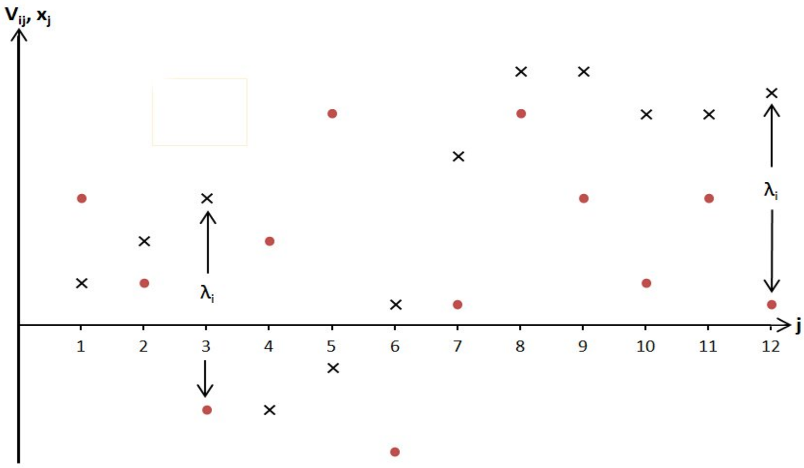
$$\forall j = 1\ldots n: \min\{x_j, \lambda + v_{ij}\} = x_j \Leftrightarrow x_j \leq \lambda + v_{ij} \Leftrightarrow \lambda \geq x_i - v_{ij}, \forall\, i = 1\ldots n$$

$$\Leftrightarrow \lambda \geq \max_j \{xj - v_{ij}\}.$$

Going back to the definition of λ_i, using (1) we get:

$$\lambda_i = \min_j \{\lambda \in R : \lambda \geq \max_j \{x_j - v_{ij}\}\}, \text{ i.e., } \lambda_i = \max_j \{x_j - v_{ij}\} = - \min \{v_{ij} - x_j\}.$$

To get a pictorial idea of what this λ_i is, consider the following:



λ_i is the maximum difference $\{x_j - v_{ij}\}$. Now, by performing the addition $\lambda \odot v_i$, we lift all coordinates of vector v_i by the amount λ_i (the minimum such amount), such that all coordinates of v_i will lie above the respective ones of x. The coordinates of x and v_i for which the maximum difference λ_i was observed will be equal in the $\lambda \odot v_i$ and x vectors, and the i will be included in the corresponding S_j's (S_3 and S_{12} above).

(2)

In math $i \in S_j$ if " x, $\lambda_i \odot v_i$ have the same j^{th} coordinate" $\Leftrightarrow x_j = \lambda_i \odot v_i|_j \Leftrightarrow$

$$x_j = \max_j \{x_j - v_{ij}\} + v_{ij} \Leftrightarrow$$

$$\Leftrightarrow x_j = \max_k \{x_k - v_{ik}\} + v_{ij} = -\min_k \{v_{ik} - x_k\} + v_{ij} = \Leftrightarrow v_{ij} - x_j = \min_k \{v_{ik} - x_k\}$$

which is exactly the condition for i to be included in S_j, in the initial definition of type (x) with respect to v.