Computationally Efficient Methods and Solutions with Least Squares Similarity Transformation Models

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Summary: Similarity 2D transformation models are presented giving emphasis on a slightly modified model, called in this study MMM, based on the general or mixed model of least-squares adjustment (Gauss-Helmert Model), where observations in both systems have different precision. For uncorrelated observations the adjustment algorithm is simplified by expressing analytically the elements of the matrices of the normal equation system. This model belongs also to the so called EIV models where solution methods, known also as TLS or WTLS, have been presented and focused in the literature the last few years. The presented model is extensively analyzed in order to become easily approachable and simple in software development. Also, for comparison reasons, the standard least-squares model (GMM) is analytically presented by its closed-form solution, to provide approximate values to the other iterative methods and because it is a familiar model in practice, e.g. in cadastral surveying, photogrammetry, GIS and image processing. Using data from four examples or experiments the presented models are compared to the published results which apply similar models.

Key words: Helmert transformation, conformal transformation, least squares models, EIV models, total least squares, weighted total least squares, modified mixed adjustment model, coordinate transformation.

1. Introduction

Similarity transformation or Helmert transformation is being widely used in a various sciences and scientific fields, such as geodesy, surveying, photogrammetry, cartography, remote sensing and GIS, e.g. Mikhail and Ackermann 1976, Paraschakis and Fotiou 1988, Ghilani and Wolf 2006, Deakin 2007. Related applications are, for instance, the datum transformation problem, e.g. the transformation of GNSS/GPS coordinates to the Transverse Mercator map projection, the connection of geodetic networks and the connection of different cadastral coordinate systems, e.g., Fotiou 2007, Fotiou and Pikridas 2012. Homogenization of printed or electronic maps produced in different geodetic reference systems (different georeferencing) and the transformation of image coordinates to map coordinates belong to the same category of applications. Of course, many other applications require different transformation models, e.g. affine, polynomial, depending on the nature of the problem and the gained experience. However, it is a matter of evaluation the choice of the proper transformation model.

In general, the problem that has to be solved is the transformation of point coordinates between coordinate systems, supposed to differ according to what describes the distortion-free model of similarity or conformal transformation.

In this study considering the model in 2-d and taking system (a) as the target system and system (b) as the start system, we suppose that their difference can be adequately described by four parameters, i.e. two translations (shifts), one scaling factor and one rotation so that the two systems are made coincident. Equivalently coordinates of any point is transformed from one system to the other one by applying the transformation parameters. Consequently, any object defined by a set of point coordinates in the start system, is shifted, rotated and uniformly scaled (resized) in order to be transformed to the target system.

The four transformation parameters are either known by a previous estimation or have to estimated or even re-estimated for testing purposes. In any case we have to deal with the determination of the transformation parameters. Data needed for this estimation is point coordinates in both systems, called common points, for at least two points. Coordinates are obtained by means of measuring processes and therefore are subjected to errors. A suitable parameter estimation method is then asked to account for inconsistencies and uncertainties of data, giving accurate results. First, it is obvious that more than two common points should be available and on the other hand a Least Squares (LS) method could be a proper and simple estimation method. Meanwhile, as it happens in many projects, there is a large amount of non-common points of the start system that have to be transformed by means of the estimated model parameters.

Apart from the functional model a stochastic model, associated with data points subjected to errors, has to be included. Using LS methods, errors are supposed to be random with zero expectation and an associated covariance matrix, known or a priori known. Moreover, in order to evaluate the model, the results of the adjustment process have to be statistically tested. Consequently, random errors are also supposed to follow the Normal or Gauss distribution and the (mathematical) model of the adjustment (functional + stochastic) has to be properly tested. Here, the random errors or the observations are considered uncorrelated, a realistic assumption in practice, although correlations are also taken into account.

The presented LS models and adjustment algorithms are also found in the literature, especially the familiar standard adjustment model where only the data points of the target system are subjected to errors while those of the start system are taken as error-free quantities (fixed). In the present study, the emphasis will be given on EIV (Errors-In-Variables) models where both data points are considered as observations. Most of the LS based solutions, that have been given focus the last few years, are based on the general mixed adjustment model of observations, known also as the Gauss-Helmert model (GHM), e.g., Jefferys 1980, Dermanis and Fotiou 1992, Schaffrin and Wieser 2008, Neitzel 2010, Simkooei and Jazaeri 2012, Sneew et al. 2015, Pan et al. 2015.

2. Problem formulation

For any point in 2D, the functional similarity transformation model is expressed by two equations, written in matrix form,

$$\begin{bmatrix} X^a \\ Y^a \end{bmatrix} = m \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} x^b \\ y^b \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \tag{2.1}$$

where (t, m, t_x, t_y) are the transformation parameters; t the rotation angle, m the scale factor and (t_x, t_y) the translations (shifts) of the start system (b) with respect to the target system (a). Applying model (2.1), coordinates (x^b, y^b) are transformed to (X^a, Y^a) as shown in Figure 1.

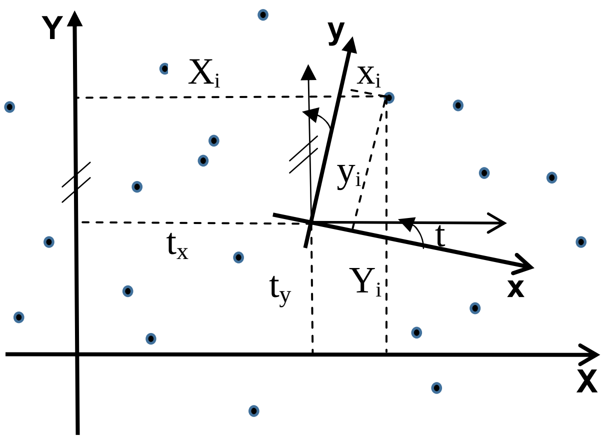

Fig. 1. *2D similarity transformation model: Data points and coordinate systems*

Introducing two independent parameters (c, d) instead of (t, m),

$$c = m\cos t, \quad d = m\sin t \qquad [m = \sqrt{c^2 + d^2}, \quad t = \arctan(d/c)] \tag{2.2}$$

model (2.1) becomes linear with respect to c and d,

$$\begin{bmatrix} X^a \\ Y^a \end{bmatrix} = \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \begin{bmatrix} x^b \\ y^b \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \tag{2.3}$$

or,

$$\begin{bmatrix} X^a \\ Y^a \end{bmatrix} = \begin{bmatrix} x^b & y^b & 1 & 0 \\ y^b & -x^b & 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \\ t_x \\ t_y \end{bmatrix} \tag{2.4}$$

According to (2.4), having two common points (n =2) we have four linear equations with four unknowns; therefore, the transformation parameters can be determined. This is a minimum requirement so that any error in the data points cannot be controlled and is absorbed by the estimated parameters. More common points (n>2) provide error control, more accurate solution and statistical testing. Applying the LS criterion, we obtain a unique optimum solution for the transformation parameters and other related estimations, all of them being of maximum accuracy (best estimations). In general, coordinates (X^a, Y^a) and (x^b, y^b) are considered as observable parameters. Special cases could be also derived as it is the next model.

3. The standard least squares model

A common case in practice accounts for errors associated only with (X^a, Y^a) while (x^b, y^b) are taken as (absolutely) known quantities (x, y). In this case, (2.4) is written as

$$\begin{bmatrix} X^a \\ Y^a \end{bmatrix} = \begin{bmatrix} x & y & 1 & 0 \\ y & -x & 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \\ t_x \\ t_y \end{bmatrix} \tag{3.1}$$

known in this form as the Gauss-Markov Model (GMM).

Substituting the observables in (2.5) with their corresponding observations and errors, i.e., $X^a = X - v_X$, $Y^a = Y - v_Y$, we have,

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} x & y & 1 & 0 \\ y & -x & 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \\ t_x \\ t_y \end{bmatrix} + \begin{bmatrix} v_X \\ v_Y \end{bmatrix} \tag{3.1}$$

or

$$\begin{bmatrix} X_1 \\ \vdots \\ X_n \\ Y_1 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ x_n & y_n & 1 & 0 \\ y_1 & -x_1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ y_n & -x_n & 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ c \\ t_x \\ t_y \end{bmatrix} + \begin{bmatrix} v_{X_1} \\ \vdots \\ v_{X_n} \\ v_{Y_1} \\ \vdots \\ v_{Y_n} \end{bmatrix} \tag{3.2}$$

In matrix notation, the linear system (3.2) is written,

$$\mathbf{y}^b = \mathbf{A}\mathbf{z}^\alpha + \mathbf{v} \tag{3.3}$$

where, $\mathbf{y}^b$ is the (2n, 1) vector of observations, $\mathbf{A}$ the (2n, 4) design matrix, $\mathbf{z}^\alpha$ the (4, 1) vector of unknown (transformation) parameters and $\mathbf{v}$ the (2n,1) vector of random errors.

In the following the LS method of observation equations or method of parameters is used and analytical closed-form solutions are presented both for equal and different precision of the observations.

3.1. Equal measurement precision

For n data points and assuming observations of equal precision, i.e., $\sigma^2_{X_i} = \sigma^2_{Y_i} = \cdots = \sigma^2$, their covariance matrix is $\mathbf{C} = \sigma^2 Q = \sigma^2 \mathbf{I}$ ($\mathbf{Q} = \mathbf{I}$) with the reference variance σ^2 known or unknown. In either case the weight matrix $\mathbf{P} = \mathbf{Q}^{-1} = \mathbf{I}$ can be used. For an unknown σ^2 (equal weights with unknown precision) an unbiased estimate has to be determined, needed for the estimation of covariance matrices of any parameters estimates. Having thus equal weights, that is $p_i = 1$, the LS solution is obtained under,

$$\sum (v^2_{X_i} + v^2_{Y_i}) = \min . \tag{3.4}$$

The well-known LS algorithm of observation equations method (e.g., Mikhail and Ackermann 1976, Dermanis and Fotiou 1992), results in the following best linear unbiased estimations $\hat{\mathbf{z}}^a$ either σ^2 is known or unknown, that is,

$$\hat{c} = \frac{\sum \tilde{x}_i X_i + \sum \tilde{y}_i Y_i}{\sum (\tilde{x}_i^2 + \tilde{y}_i^2)} \qquad \hat{d} = \frac{\sum \tilde{y}_i X_i - \sum \tilde{x}_i Y_i}{\sum (\tilde{x}_i^2 + \tilde{y}_i^2)} \tag{3.5}$$

$$\hat{t}_x = \hat{s}_{\tilde{x}} - (\hat{c}\overline{x} + \hat{d}\overline{y}) = \overline{X} - \hat{c}\overline{x} - \hat{d}\overline{y}, \quad \hat{t}_y = \hat{s}_{\tilde{y}} - (-\hat{d}\overline{x} + \hat{c}\overline{y}) = \overline{Y} + \hat{d}\overline{x} - \hat{c}\overline{y} \tag{3.6}$$

where, $(\tilde{x}, \tilde{y})$ are the reduced (x, y) coordinates of the start system to their centroid $(\bar{x}, \bar{y})$ computed by

$$\tilde{x}_i = x_i - \bar{x}, \quad \tilde{y}_i = y_i - \bar{y}, \qquad \bar{x} = \frac{\sum x_i}{n}, \quad \bar{y} = \frac{\sum y_i}{n} \tag{3.7}$$

and $(\hat{s}_{\tilde{x}}, \hat{s}_{\tilde{y}})$ the translations of the reduced system, equal to the centroid $(\bar{X}, \bar{Y})$. We remind that centroid reduction leads to a diagonal structure of the normal equation matrix as the sum of such reduced coordinates is always zero.

In addition, the estimations of errors, observables and the a-posteriori variance, are given by

$$\hat{v}_{X_i} = X_i - (\hat{c}x_i + \hat{d}y_i + \hat{t}_x), \qquad \hat{v}_{Y_i} = Y_i - (-\hat{d}x_i + \hat{c}y_i + \hat{t}_y) \tag{3.8}$$

$$\hat{X}_i = X_i - \hat{v}_{X_i} = (\hat{c}x_i + \hat{d}y_i + \hat{t}_x), \quad \hat{Y}_i = Y_i - \hat{v}_{Y_i} = (-\hat{d}x_i + \hat{c}y_i + \hat{t}_y) \tag{3.9}$$

$$\hat{\sigma}^2 = \frac{\sum(\hat{v}_{X_i}^2 + \hat{v}_{Y_i}^2)}{2n-4} \tag{3.10}$$

where the posteriori variance can be used instead of an unknown σ^2 and be statistically tested against an priori σ_o^2 .

Note that estimations of the scale factor and the rotation angle $(\hat{m}, \hat{t})$ can be computed by (2.2), realizing that the rotation angle as given by the inverse tangent function ($-\pi/2 \leq t \leq +\pi/2$) should be reduced to the correct quadrant, e.g. considering positive counterclockwise ($0 \leq t < 2\pi$).

The precision of the estimated parameters is given by the respective variances or standard deviations, that is,

$$\hat{\sigma}_{\hat{c}}^2 = \sigma^2 \frac{1}{\sum(\tilde{x}_i^2 + \tilde{y}_i^2)} \qquad \hat{\sigma}_{\hat{d}}^2 = \hat{\sigma}_{\hat{c}}^2 \tag{3.11}$$

$$\hat{\sigma}_{\hat{t}_x}^2 = \sigma^2 \left(\frac{\bar{x}^2 + \bar{y}^2}{\sum(\tilde{x}_i^2 + \tilde{y}_i^2)} + \frac{1}{n} \right) \qquad \hat{\sigma}_{\hat{t}_y}^2 = \hat{\sigma}_{\hat{t}_x}^2 \tag{3.12}$$

Note again that the posteriori variance is used in case of an unknown σ^2 .

Also, the transformed (x^s, y^s) coordinates of any non-common point (x, y) of the start system is given by

$$\begin{bmatrix} x^s \\ y^s \end{bmatrix} = \begin{bmatrix} \hat{c} & \hat{d} \\ -\hat{d} & \hat{c} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \hat{t}_x \\ \hat{t}_y \end{bmatrix} \tag{3.13}$$

Applying the law of error propagation, precision measures can be computed for any estimation, e.g. for the transformed coordinates.

Sometimes, for some reason, distances (scale, size of objects) between any pair of points should be preserved, constraining thus m=1 and performing a so called rigid transformation (only rotation and translation). This condition can be fulfilled after the estimation of four transformation parameters, rearranging the estimated parameters and realizing that the rotation is the same, i.e., $m = 1, t' = t = \arctan(\hat{d} / \hat{c})$, $c' = \cos t, d' = \sin t$, $\hat{t}'_x = \overline{X} - (c'\overline{x} + d'\overline{y})$, $\hat{t}'_y = \overline{Y} - (-d'\overline{x} + c'\overline{y})$. With these new parameter estimations, other estimations are computed, following the adjustment algorithm. Similar conditions, as for example considering zero rotation or zero translation, can be easily treated.

In geodesy and surveying it is a common practice to keep unaltered the coordinates of common points in the target system, when they are e.g. control points of a geodetic datum, although estimations of errors for these points have been computed. In doing so, we have a best fit of the start system to the target system and the errors just give a measure of the fit or a measure of assessment of the used transformation model.

3.2. Different measurement precision

Considering different precision for each observation (X_i, Y_i), we have, $\sigma^2_{X_1} \neq \cdots \neq \sigma^2_{X_n} \neq \sigma^2_{Y_1} \neq \cdots \neq \sigma^2_{Y_n}$. The reduction of (x_i, y_i) to the weighted centroid does not lead to a diagonal normal equation matrix and a simple closed-form solution; therefore, the solution is carried out by inverting a full (4,4) normal equation matrix. A simplification occurs when the same precision or weight is associated with the coordinates (X_i, Y_i) of each point, so that $\sigma^2_{X_i} = \sigma^2_{Y_i} = \sigma^2_i$.
Actually, $\sigma^2_i = \sigma^2 q^2_i$, with σ^2 a reference variance, known or unknown. Incorporating weights in this approach, $\mathbf{C} = \sigma^2 \mathbf{Q}$, where $\mathbf{Q} = \mathrm{diag}(q^2_1, ..., q^2_n, q^2_1, ..., q^2_n)$,
and $\mathbf{P} = \mathbf{Q}^{-1} = \mathrm{diag}(1/q^2_1, .., 1/q^2_n, 1/q^2_1, .., 1/q^2_n) = \mathrm{diag}(p_1, .., p_n, p_1, .., p_n)$.
Applying again the LS criterion with weights,

$$\sum p_i (v^2_{X_i} + v^2_{Y_i}) = \min \tag{3.14}$$

and performing the centroid reduction of (x_i, y_i), the LS estimations are given by the following simple again expressions, comparable to the above (3.5) and (3.6),

$$\hat{c} = \frac{\sum p_i (\tilde{x}_i X_i + \tilde{y}_i Y_i)}{\sum p_i (\tilde{x}^2_i + \tilde{y}^2_i)} \qquad \hat{d} = \frac{\sum p_i (\tilde{y}_i X_i - \tilde{x}_i Y_i)}{\sum p_i (\tilde{x}^2_i + \tilde{y}^2_i)} \tag{3.15}$$

$$\hat{t}_x = \hat{s}_{\tilde{x}} - (\hat{c}\overline{x} + \hat{d}\overline{y}) = \overline{X} - \hat{c}\overline{x} - \hat{d}\overline{y}, \quad \hat{t}_y = \hat{s}_{\tilde{y}} - (-\hat{d}\overline{x} + \hat{c}\overline{y}) = \overline{Y} + \hat{d}\overline{x} - \hat{c}\overline{y} \tag{3.16}$$

where, $(\tilde{x}, \tilde{y})$ are the reduced coordinates of the start system to their weighted centroid $(\overline{x}, \overline{y})$ given by

$$\tilde{x}_i = x_i - \bar{x}, \quad \tilde{y}_i = y_i - \bar{y} \qquad \bar{x} = \frac{\sum p_i x_i}{\sum p_i}, \quad \bar{y} = \frac{\sum p_i y_i}{\sum p_i} \tag{3.17}$$

and $(\hat{s}_{\tilde{x}}, \hat{s}_{\tilde{y}})$ the translations of the reduced system, equal to the weighted centroid $(\bar{X} = \sum p_i X_i / \sum p_i, \bar{Y} = \sum p_i Y_i / \sum p_i)$. In a similar manner, the estimations of errors and observables are computed as above from (3.8, 3.10) while the a posteriori variance is given by

$$\hat{\sigma}^2 = \frac{\sum p_i (\hat{v}_{X_i}^2 + \hat{v}_{Y_i}^2)}{2n - 4} \tag{3.18}$$

In addition, precision estimates for the transformation parameters are obtained by

$$\hat{\sigma}_{\hat{c}}^2 = \sigma^2 \frac{1}{\sum p_i (\tilde{x}_i^2 + \tilde{y}_i^2)} \qquad \hat{\sigma}_{\hat{d}}^2 = \hat{\sigma}_{\hat{c}}^2 \tag{3.19}$$

$$\hat{\sigma}_{\hat{t}_x}^2 = \sigma^2 \left(\frac{\bar{x}^2 + \bar{y}^2}{\sum p_i (\tilde{x}_i^2 + \tilde{y}_i^2)} + \frac{1}{\sum p_i} \right) \qquad \hat{\sigma}_{\hat{t}_y}^2 = \hat{\sigma}_{\hat{t}_x}^2 \tag{3.20}$$

In the literature and particularly in cadastral surveys and GIS (e.g., Deakin 2007), the presented algorithm is applied with a different interpretation of the observation errors, where they supposed to consist of two parts; one part related to the observation errors in the target system and the other to the transformed errors of the start system. Though this is a reasonable hypothesis and practically workable, a rigorous treatment demands for a different model, presented below, as coordinates on both systems are subjected to errors.

4. The modified mixed model of adjustment

Now we come up to the point that both (X^a, Y^a), (x^b, y^b) are observable parameters and (X, Y), (x, y) the corresponding observations. It is evident that the functional model is a mixed-type model and should be treated according to the general method of LS, also called Total LS (TLS) and Weighted TLS (WTLS), e.g. Rossikopoulos and Fotiou 1993. In addition, this model belongs to the so called EIV (Errors In Variables) models that have been in the spotlight the last few years. Actually, we have to do with an adjustment model of observations of mixed equations, known traditionally in geodetic community as a Gauss - Helmert Model (GHM). Recently focus has been given on a slight extension of GHM, called here MMM (Modified Mixed Model), e.g, Simkooei and Jazaeri 2012, Pan et al. 2015, Fotiou 2017.

4.1. The modified mixed model of the similarity transformation

According to the LS method of mixed equations, any common point gives two (non-linear) mixed equations, in matrix form,

$$\begin{bmatrix} X_i^a \\ Y_i^a \end{bmatrix} = \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \begin{bmatrix} x_i^b \\ y_i^b \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \tag{4.1}$$

or

$$\begin{bmatrix} f_i \\ g_i \end{bmatrix} = \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \begin{bmatrix} x_i^b \\ y_i^b \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} - \begin{bmatrix} X_i^a \\ Y_i^a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad i = 1,\dots, n. \tag{4.2}$$

Following the linearization process, we expand (4.2) in a Taylor series up to first order terms, around the approximate values (c^o, d^o, t_x^o, t_y^o) for the unknown parameters and $(X_i^o, Y_i^o, x_i^o, y_i^o)$ for the unknown observables. We underline that the modification or a slight extend in MMM, starts with the series expansion around the approximate point (rigorous theory) and not around the observed point (X_i, Y_i, x_i, y_i) as used in GHM. In many cases the difference between the two models is practically insignificant.

It follows that, if $\mathbf{F}_i = \begin{bmatrix} f_i & g_i \end{bmatrix}^T$, the Taylor series is written as,

$$\mathbf{F}_i = \mathbf{F}_i^o + \left.\frac{\partial \mathbf{F}_i}{\partial c}\right|_o (c - c^o) + \left.\frac{\partial \mathbf{F}_i}{\partial d}\right|_o (d - d^o) + \left.\frac{\partial \mathbf{F}_i}{\partial t_x}\right|_o (t_x - t_x^o) + \left.\frac{\partial \mathbf{F}_i}{\partial t_y}\right|_o (t_y - t_y^o) +$$

$$+ \left.\frac{\partial \mathbf{F}_i}{\partial X_i^a}\right|_o (X_i^a - X_i^o) + \left.\frac{\partial \mathbf{F}_i}{\partial Y_i^a}\right|_o (Y_i^a - Y_i^o) + \left.\frac{\partial \mathbf{F}_i}{\partial x_i^b}\right|_o (x_i^b - x_i^o) + \left.\frac{\partial \mathbf{F}_i}{\partial y_i^b}\right|_o (y_i^b - y_i^o) + .. = \mathbf{0} \tag{4.3}$$

Understanding that

$$c = c^o + \delta c, \qquad d = d^o + \delta d, \qquad t_x = t_x^o + \delta t_x, \quad t_y = t_y^o + \delta t_y, \tag{4.4}$$

$$X_i^a = X_i - v_{X_i}, \quad Y_i^a = Y_i - v_{Y_i}, \quad x_i^b = x_i - v_{x_i}, \quad y_i^b = y_i - v_{y_i}, \tag{4.5}$$

$$X_i^o = X_i - v_{X_i}^o, \quad Y_i^o = Y_i - v_{Y_i}^o, \quad x_i^o = x_i - v_{x_i}^o, \quad y_i^o = y_i - v_{y_i}^o, \tag{4.6}$$

$$\mathbf{F}_i^o = \begin{bmatrix} f_i^o \\ g_i^o \end{bmatrix} = \begin{bmatrix} c^o x_i^o + d^o y_i^o + t_x^o - X_i^o \\ -d^o x_i^o + c^o y_i^o + t_y^o - Y_i^o \end{bmatrix} \tag{4.7}$$

and accounting for the partial derivatives, after some arrangements, (4.3) becomes

$$\begin{bmatrix} c^o x_i + d^o y_i + t_x^o - X_i \\ -d^o x_i + c^o y_i + t_y^o - Y_i \end{bmatrix} + \begin{bmatrix} (x_i - v_{x_i}^o) & (y_i - v_{y_i}^o) & 1 & 0 \\ (y_i - v_{y_i}^o) & -(x_i - v_{x_i}^o) & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta c \\ \delta d \\ \delta t_x \\ \delta t_y \end{bmatrix} +$$

$$- \begin{bmatrix} -1 & 0 & c^o & d^o \\ 0 & -1 & -d^o & c^o \end{bmatrix} \begin{bmatrix} v_{X_i} \\ v_{Y_i} \\ v_{x_i} \\ v_{y_i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.8}$$

In a detailed matrix structure, (4.8) is written as,

$$\begin{bmatrix} c^o x_1 + d^o y_1 + t_x^o - X_1 \\ \vdots \\ c^o x_n + d^o y_n + t_x^o - X_n \\ -d^o x_1 + c^o y_1 + t_y^o - Y_1 \\ \vdots \\ -d^o x_n + c^o y_n + t_y^o - Y_n \end{bmatrix} + \begin{bmatrix} (x_1 - v_{x_1}^o) & (y_1 - v_{y_1}^o) & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ (x_n - v_{x_n}^o) & (y_n - v_{y_n}^o) & 1 & 0 \\ (y_1 - v_{y_1}^o) & -(x_1 - v_{x_1}^o) & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ (y_n - v_{yn}^o) & -(x_n - v_{x_n}^o) & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta c \\ \delta d \\ \delta t_x \\ \delta t_y \end{bmatrix} +$$

$$- \begin{bmatrix} -\mathbf{I}_n & \mathbf{0}_n & c^o \mathbf{I}_n & d^o \mathbf{I}_n \\ \mathbf{0}_n & -\mathbf{I}_n & -d^o \mathbf{I}_n & c^o \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{v}_X \\ \mathbf{v}_Y \\ \mathbf{v}_x \\ \mathbf{v}_y \end{bmatrix} = \mathbf{0}_{2n,1} \tag{4.9}$$

where,

$$\mathbf{v}_X = \begin{bmatrix} v_{X_1} & \cdots & v_{X_n} \end{bmatrix}^T, \quad \mathbf{v}_Y = \begin{bmatrix} v_{Y_1} & \cdots & v_{Y_n} \end{bmatrix}^T \tag{4.10}$$

$$\mathbf{v}_x = \begin{bmatrix} v_{x_1} & \cdots & v_{x_n} \end{bmatrix}^T, \quad \mathbf{v}_y = \begin{bmatrix} v_{y_1} & \cdots & v_{y_n} \end{bmatrix}^T \tag{4.11}$$

or, in brief

$$\mathbf{w} + \mathbf{A}\mathbf{z} - \mathbf{B}\mathbf{v} = \mathbf{0} \tag{4.12}$$

Model (4.12) is the linear system of the general adjustment model with a slight modification in the design matrix **A**, where the first two columns depend on errors of the observations of the start system while matrix **B** is unaffected.

The LS solution is then obtained using the method of Lagrange multipliers, i.e.,

$$\sum \left(p_{X_i} v_{X_i}^2 + p_{Y_i} v_{Y_i}^2 + p_{x_i} v_{x_i}^2 + p_{y_i} v_{y_i}^2 \right) = \mathbf{v}^T \mathbf{P} \mathbf{v} = \min, \quad \text{under} \quad \mathbf{F}_i = \mathbf{0} \tag{4.13}$$

The adjustment algorithm is applied through an iterative process, where the initial approximate values (c^o, d^o, t_x^o, t_y^o) and $(v_{x_i}^o, v_{y_i}^o)$ are properly updated until convergence is achieved. (4.14)

In order to facilitate the computations and/or the development of a software, a few details for the implementation of the algorithm are given below.
First, we form the (diagonal) covariance and weight matrices,

$$\mathbf{C} = \sigma^2 \mathbf{Q} = \sigma^2 \mathrm{diag}(\mathbf{Q}_X, \mathbf{Q}_Y, \mathbf{Q}_x, \mathbf{Q}_y) \tag{4.15}$$

$$\mathbf{P} = \mathbf{Q}^{-1} = \mathrm{diag}(\mathbf{Q}_X^{-1}, \mathbf{Q}_Y^{-1}, \mathbf{Q}_x^{-1}, \mathbf{Q}_y^{-1}) = \mathrm{diag}(\mathbf{P}_X, \mathbf{P}_Y, \mathbf{P}_x, \mathbf{P}_y) \tag{4.16}$$

where σ^2 is known or unknown, and $\hat{\sigma}^2$ is used instead of an unknown one.
The normal equation system is then formed by,

$$(\mathbf{A}^T \mathbf{M}^{-1} \mathbf{A})\mathbf{z} = (\mathbf{A}^T \mathbf{M}^{-1} \mathbf{w}) \quad \text{or} \quad \mathbf{Nz} = \mathbf{-u} \tag{4.17}$$

where the (2n, 2n) symmetric matrix **M** and the (2n, 1) column matrix **w** are,

$$\mathbf{M} = \mathbf{B}\mathbf{P}^{-1}\mathbf{B}^T = \begin{bmatrix} \sigma_{f_1}^2 & & & \sigma_{f_1 g_1} & & \\ & \ddots & & & \ddots & \\ & & \sigma_{f_n}^2 & & & \sigma_{f_n g_n} \\ \sigma_{f_1 g_1} & & & \sigma_{g_1}^2 & & \\ & \ddots & & & \ddots & \\ & & \sigma_{f_n g_n} & & & \sigma_{g_n}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_f & \mathbf{C}_{fg} \\ \mathbf{C}_{fg} & \mathbf{C}_g \end{bmatrix} \tag{4.18}$$

$$\mathbf{w} = \begin{bmatrix} c^o x_1 + d^o y_1 + t_x^o - X_1 \\ \vdots \\ c^o x_n + d^o y_n + t_x^o - X_n \\ -d^o x_1 + c^o y_1 + t_y^o - Y_1 \\ \vdots \\ -d^o x_n + c^o y_n + t_y^o - Y_n \end{bmatrix} = \begin{bmatrix} \mathbf{w}_f \\ \mathbf{w}_g \end{bmatrix} \tag{4.19}$$

Matrix **M** consists of four diagonal (n, n) submatrices facilitating its analytic inversion by means of well-known algorithms for partitioned matrix structures. The elements of **M** are then given by,

$$\sigma_{f_i}^2 = \frac{c^{o2}}{p_{x_i}} + \frac{d^{o2}}{p_{y_i}} + \frac{1}{p_{X_i}}, \quad \sigma_{g_i}^2 = \frac{d^{o2}}{p_{x_i}} + \frac{c^{o2}}{p_{y_i}} + \frac{1}{p_{Y_i}}, \quad \sigma_{f_i g_i}^2 = \frac{-c^o d^o}{p_{x_i}} + \frac{c^o d^o}{p_{y_i}} \tag{4.20}$$

and the inverse $\mathbf{M}^{-1}$ is formed as,

$$\mathbf{M}^{-1} = \begin{bmatrix} \frac{\sigma^2_{g_1}}{r_1} & & -\frac{\sigma^2_{f_1g_1}}{r_1} & \\ & \ddots & & \ddots \\ & \frac{\sigma^2_{g_n}}{r_n} & & -\frac{\sigma^2_{f_1g_1}}{r_1} \\ -\frac{\sigma^2_{f_1g_1}}{r_1} & & \frac{\sigma^2_{f_1}}{r_1} & \\ & \ddots & & \ddots \\ & -\frac{\sigma^2_{f_1g_1}}{r_1} & & \frac{\sigma^2_{f_n}}{r_n} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{gr} & -\mathbf{C}_{fgr} \\ -\mathbf{C}_{fgr} & \mathbf{C}_{fr} \end{bmatrix} \tag{4.21}$$

where, $r_i = \sigma^2_{f_i}\sigma^2_{g_i} - (\sigma_{f_ig_i})^2$.

In addition, the elements of matrices **N**, **u** of the normal equation system, from which the LS solution is obtained, are given analytically, based on the above structure. Putting,

$$bx(i) = (x_i - v^o_{x_i}),\ by(i) = (y_i - v^o_{y_i}),\ bxy(i) = (x_i - v^o_{x_i})(y_i - v^o_{y_i}) \tag{4.22}$$

the elements of **N**,

$$\mathbf{N} = \mathbf{A}^T\mathbf{M}^{-1}\mathbf{A} = \begin{bmatrix} N_{11} & N_{12} & N_{13} & N_{14} \\ & N_{22} & N_{23} & N_{24} \\ & & N_{33} & N_{34} \\ \text{sym.} & & & N_{44} \end{bmatrix} \tag{4.23}$$

are derived, i.e.,

$$N_{11} = \sum\{(bx(i)^2\sigma^2_{g_i} + by(i)^2\sigma^2_{f_i} - 2bxy(i)\sigma_{f_ig_i})/r_i\}$$

$$N_{12} = \sum\{[bxy(i)(\sigma^2_{g_i} - \sigma^2_{f_i}) + (bx(i)^2 - by(i)^2)\sigma_{f_ig_i})]/r_i\}$$

$$N_{13} = \sum\{(bx(i)\sigma^2_{g_i} - by(i)\sigma_{f_ig_i})/r_i\}, \quad N_{14} = \sum\{(by(i)\sigma^2_{f_i} - bx(i)\sigma_{f_ig_i})/r_i\}$$

$$N_{22} = \sum\{(by(i)^2\sigma^2_{g_i} + bx(i)^2\sigma^2_{f_i} + 2bxy(i)\sigma_{f_ig_i})/r_i\}$$

$$N_{23} = \sum\{(by(i)\sigma^2_{g_i} + bx(i)\sigma_{f_ig_i})/r_i\}, \quad N_{24} = \sum\{(-by(i)\sigma_{f_ig_i} - bx(i)\sigma^2_{f_i})/r_i\}$$

$$N_{33} = \sum(\sigma^2_{g_i}/r_i), \quad N_{34} = \sum(-\sigma_{f_ig_i}/r_i), \quad N_{44} = \sum(\sigma^2_{f_i}/r_i)$$

Moreover, the elements of **u**,

$$\mathbf{u} = \mathbf{A}^T\mathbf{M}^{-1}\mathbf{w} = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}^T \tag{4.24}$$

are derived by,

$$u_1 = \sum\{[w_{f_i}(x_i\sigma^2_{g_i} - y_i\sigma_{f_ig_i}) + w_{g_i}(-x_i\sigma_{f_ig_i} + y_i\sigma^2_{f_i})]/r_i\}$$

$$u_2 = \sum\{[w_{f_i}(y_i\sigma^2_{g_i} + x_i\sigma_{f_ig_i}) + w_{g_i}(-y_i\sigma_{f_ig_i} - x_i\sigma^2_{f_i})]/r_i\}$$

$$u_3 = \sum\{(w_{f_i}\sigma^2_{g_i} + w_{g_i}\sigma_{f_ig_i})/r_i\}, \quad u_4 = \sum\{(-w_{f_i}\sigma_{f_ig_i} + w_{g_i}\sigma^2_{f_i})/r_i\}$$

Next the corrections of parameter estimations are given by,

$$\hat{\mathbf{z}} = \begin{bmatrix} \delta\hat{c} & \delta\hat{d} & \delta\hat{t}_x & \delta\hat{t}_y \end{bmatrix}^T = -\mathbf{N}^{-1}\mathbf{u} \tag{4.25}$$

and the parameter estimates by

$$\hat{\mathbf{z}}^a = \begin{bmatrix} \hat{c} & \hat{d} & \hat{t}_x & \hat{t}_y \end{bmatrix}^T = \mathbf{z}^o + \hat{\mathbf{z}} = \begin{bmatrix} c^o + \delta\hat{c} & d^o + \delta\hat{d} & t^o_x + \delta\hat{t}_x & t^o_y + \delta\hat{t}_y \end{bmatrix}^T \tag{4.26}$$

understanding that an inversion algorithm, e.g. Cholesky decomposition, is needed for $\mathbf{N}^{-1}$. Note that the elements of $\mathbf{N}$ could be very large and/or very small, the matrix inversion algorithm should be tested by double precision arithmetic.

Going on with the adjustment algorithm, the estimation of errors and observable parameters (adjusted observations) are,

$$\hat{\mathbf{v}} = \begin{bmatrix} \hat{\mathbf{v}}_X \\ \hat{\mathbf{v}}_Y \\ \hat{\mathbf{v}}_\mathbf{x} \\ \hat{\mathbf{v}}_\mathbf{y} \end{bmatrix} = \mathbf{P}^{-1}\mathbf{B}^T\mathbf{M}^{-1}(\mathbf{w} + \mathbf{A}\hat{\mathbf{z}}) \tag{4.27}$$

$$\begin{bmatrix} \hat{X}_i \\ \hat{Y}_i \end{bmatrix} = \begin{bmatrix} X_i \\ Y_i \end{bmatrix} - \begin{bmatrix} \hat{v}_{X_i} \\ \hat{v}_{Y_i} \end{bmatrix}, \quad \begin{bmatrix} \hat{x}_i \\ \hat{y}_i \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \end{bmatrix} - \begin{bmatrix} \hat{v}_{x_i} \\ \hat{v}_{y_i} \end{bmatrix} \tag{4.28}$$

For the error estimations, and taking,

$$cf(i) = w_{f_i} + df_i = (c^o x_1 + d^o y_1 + t^o_x - X_1) + (bx(i)\delta\hat{c} + by(i)\delta\hat{d} + \delta\hat{t}_x) \tag{4.29}$$

$$cg(i) = w_{g_i} + dg_i = (-d^o x_i + c^o y_i + t^o_y - Y_i) + (-bx(i)\delta\hat{d} + by(i)\delta\hat{c} + \delta\hat{t}_y) \tag{4.30}$$

we reach at the analytic expressions,

$$\hat{v}_{X_i} = [(-cf(i)\sigma^2_{g_i} + cg(i)\sigma_{f_ig_i})/r_i]/p_{X_i} \tag{4.31a}$$

$$\hat{v}_{Y_i} = [(cf(i)\sigma_{f_i g_i} - cg(i)\sigma_{f_i}^2)/r_i]/p_{Y_i} \tag{4.31b}$$

$$\hat{v}_{x_i} = [(c^o\sigma_{g_i}^2 + d^o\sigma_{f_i g_i})cf(i)/r_i]/p_{x_i} + [(-c^o\sigma_{f_i g_i} - d^o\sigma_{f_i}^2)cg(i)/r_i]/p_{x_i} \tag{4.32a}$$

$$\hat{v}_{y_i} = [(d^o\sigma_{g_i}^2 - c^o\sigma_{f_i g_i})cf(i)/r_i]/p_{y_i} + [(-d^o\sigma_{f_i g_i} + c^o\sigma_{f_i}^2)cg(i)/r_i]/p_{y_i} \tag{4.32b}$$

Also, the estimation of an unknown variance factor, i.e. the posteriori variance, is

$$\hat{\sigma}^2 = \frac{\sum\left(p_{X_i}\hat{v}_{X_i}^2 + p_{Y_i}\hat{v}_{Y_i}^2 + p_{x_i}\hat{v}_{x_i}^2 + p_{y_i}\hat{v}_{y_i}^2\right)}{4n-4} =$$

$$= \frac{\hat{\mathbf{v}}_X^T\mathbf{P}_X\hat{\mathbf{v}}_X + \hat{\mathbf{v}}_Y^T\mathbf{P}_Y\hat{\mathbf{v}}_Y + \hat{\mathbf{v}}_x^T\mathbf{P}_x\hat{\mathbf{v}}_x + \hat{\mathbf{v}}_y^T\mathbf{P}_y\hat{\mathbf{v}}_y}{4n-4} \tag{4.33}$$

From the covariance matrix of the transformation parameters, which is the inverse $\sigma^2\mathbf{N}^{-1}$, precision measures can be computed as well, e.g. the diagonal elements express the variances of the parameter estimates.

The implementation of the MMM algorithm start with initial approximate values (c^o, d^o, t_x^o, t_y^o) computed by a suitable way, preferably by the standard LS solution presented above in chapter 3 with equal observation precision. In parallel, the initial values for the approximate errors in matrix **A** are taken as zero $(v_{x_i}^o = 0, v_{y_i}^o = 0)$. In this way, the estimations of corrections $\delta\hat{c}$, $\delta\hat{d}, \delta\hat{t}_x, \delta\hat{t}_y$ and $\hat{v}_{x_i}, \hat{v}_{y_i}$ are obtained. In the next iteration new approximate values are used, as derived by the previous solution and new estimates $\delta\hat{c}^{(1)}$, $\delta\hat{d}^{(1)}, \delta\hat{t}_x^{(1)}, \delta\hat{t}_y^{(1)}$ and $\hat{v}_{x_i}^{(1)}, \hat{v}_{y_i}^{(1)}$ are again obtained. With these better values, the second iteration starts and so on until a convergence is achievement.

It should be noted that within each iteration, though it is not necessary, other updated estimates, such as $\hat{X}_i^{(k)} = X_i - \hat{v}_{X_i}^{(k)}$, $\hat{Y}_{ii}^{(k)} = Y_i - \hat{v}_{Y_i}^{(k)}$ and $\hat{\sigma}^{2(k)}$, could be also obtained, instead of the end of the whole process, resulting in their final adjusted values.

Convergence criterions for the successive absolute differences are set, usually for the updated transformation parameters or even for all the updated parameters. The threshold depends on the degree of closeness of the initial approximate values to their best values and on the level of accuracy needed. For example, if the observations are UTM map coordinates given with ten significant figures and of mm-precision, a threshold ε_c = 1.0E-10 to 1.0E-12 for c and d, i.e., $|\hat{c}^{(i+1)} - \hat{c}^{(i)}| \leq \varepsilon_c$, and ε_t = 1.0E-03m to 1.0E-05m for the translations, i.e., $|\hat{t}_x^{(i+1)} - \hat{t}_x^{(i)}| \leq \varepsilon_t$, could be an adequate choice. A good practice is to use double precision arithmetic and round properly at the end of the whole process.

The above estimations are Best Linear Unbiased Estimations (BLUE) according to the LS principles. A statistical evaluation of the model, could be a global test of the variance and data snooping for detection of outliers. In many practical applications, without demanding high accuracy, instead of statistical hypothesis testing for outliers a marginal value/threshold, e.g. 3 to 5 times the precision of the observations could be set. The fact that matrix **A** depends on observation errors does not have a significant impact on the estimation of covariance matrices.

A very good approximation to the above rigorous MMM solution is given by the GHM as traditionally applied. The only difference with the presented MMM algorithm is that the design matrix **A** in the GHM depends on (x_i, y_i) and not on $(x_i^o = x_i - v_{x_i}^o)$, $(y_i^o = y_i - v_{y_i}^o)$, so that **A** remains constant and the transformation parameters are the only ones that are updated.

The similarity transformation adjustment algorithm with MMM covers also the case with equal precision as a special case, but it is preferable to use the above closed-form solution. Using the GHM with observations of equal precision the parameter estimations is independent of their approximate values and are identical to those derived by the standard LS adjustment (GMM). The same holds if the precision of the observations is the same for each coordinate system but different between the two systems (e.g. Dermanis and Fotiou 1992).

In this study the issue with the transformation of the non-common points, using MMM or GHM model is not discussed. However, it should be realized that in a rigorous transformation, the non-common points could be correlated to the common ones and their transformation should depend on the precision of the estimated parameters (e.g., Fotiou and Rossikopoulos 1993, Kaltsikis et al. 1994).

5. Examples and comparison of the results

The LS iterative MMM algorithm is easily understood and implemented by means of a software created and/or adapted to particular needs. Moreover, the traditional LS models could be included as special cases. Especially, when the MMM or the traditional GHM is used, the standard GMM can provide approximate values for the transformation parameters; on the other hand, the number of iterations against unsatisfying initial values is almost minimized.

In this study, a Fortran program, written in 'Simply Fortran environment', was created and tested using data of numerical examples taken from the literature. In the following the results, obtained by means of the above mixed modified model and the traditional models, are presented and compared with the published results.

Four examples are presented in tabular form to facilitate reading and comparison. In all examples both sets of coordinates are observations with equal (example 1 and 4) or different precision (example 2 and 3). In addition, results from the standard GM model are given for comparison reasons and as a means to provide approximate values to the other models.

In Tables 1 to 3, and Tables 4 to 6 two examples are presented whose data have been taken from Neitzel (2010). In the first example data have equal precision (p_i=1) and the results are almost identical among the presented models, the only difference being in the number of iterations. Having equal weights, as also happens in example 4, it is verified the theoretical conclusion that the standard GMM model gives identical solution with that of GHM and MMM except the error estimates and whatever is related to those, as naturally expected since in the GMM only one set of data points is subjected to errors.

***Table 1**: Observations of equal precision*

Example 1: Data taken from Neitzel (2010)				
Point	Target system (a)		Start system (b)	
	X(mm)	Y(mm)	x(mm)	y(mm)
1	-117.478	0	17.856	144.794
2	117.472	0	252.637	154.448
3	0.015	-117.410	140.089	32.326
4	-0.014	117.451	130.400	267.027
Point	Weights			
	p_X	p_Y	p_x	p_y
i	1	1	1	1

***Table 2**. Models and results of Example 1*

Parameters	Neitzel (2010)	This study		
	'Iterated linearized GHM'	Modified Mixed Model (MMM)	Traditional GHM	Standard GMM
$\hat{c}$	0.99900748078	0.99900746914	0.99900746914	0.99900746914
$\hat{d}$	-0.04109806319*	0.04109806272	0.04109806272	0.04109806272
$\hat{t}_x$ (mm)	-141.2628	-141.2628	-141.2628	-141.2628
$\hat{t}_y$ (mm)	-143.9316	-143.9316	-143.9316	-143.9316
$\hat{m}$	0.99985248784	0.99985247619	0.99985247619	0.99985247619
$\hat{t}$ (= -t)	-2.3557567*			
$\hat{t}$ (0≤t<360º)		2.3557567	2.3557567	2.3557567
$\hat{v}^T P \hat{v}$	0.000643	0.000643	0.000643	0.001286
$\hat{\sigma}^2$	0.0001608 ?	0.000054	0.000054	0.000097
approx. values from Neitzel	'several' iterations	3 iter.	3 iter.	-
approx. values from GMM	not given	0 iter.	0 iter.	-

Table 3. *Estimation of errors (mm) of Example 1*

Neitzel (2010) / This study(MMM)				
Point	$\hat{v}_X$	$\hat{v}_Y$	$\hat{v}_x$	$\hat{v}_y$
1	-0.0021 -0.0021	0.0076 0.0076	0.0024 0.0024	-0.0075 -0.0075
2	0.0005 0.0005	0.0099 0.0099	-0.0001 -0.0001	-0.0099 -0.0099
3	-0.0004 -0.0004	-0.0074 -0.0074	-0.0000 0.0000	0.0075 0.0075
4	-0.0020? 0.0020	-0.0101 -0.0101	-0.0024 -0.0024	0.0100 0.0100

In the second example (Tables 4 to 6) with different data precision, some differences occur related to the transformation parameters, especially to the translations. However, it is remarkable to see that the error estimates are almost identical and the differences are less than one mm, meaning that the two apparently different sets of parameters are consistent or practically equivalent. It should be underlined that the order of magnitude of the coordinates in relation to the used weights have a direct impact on the magnitude of the elements of the normal equation matrix (large difference among them) and on its inversion. We have tried a solution with reduced coordinates but the solution did not change.

Table 4. *Observations of different precision*

Example 2: Data taken from Neitzel (2010)				
Point	Target system (a)		Start system (b)	
	X(m)	Y(m)	x(m)	y(m)
3	4540134.2780	382379.8964	4540124.0904	382385.9980
185	4539937.3890	382629.7872	4539927.2250	382635.8691
2796	4539979.7390	381951.4785	4539969.5670	381957.5705
2996	4540326.4610	381895.0089	4540316.2940	381901.0932
5005	4539216.3870	382184.4352	4539206.2110	382190.5278
Point	Weights			
	p_X	p_Y	p_x	p_y
3	10.0000	14.2857	5.8824	12.5000
185	0.8929	1.4286	0.9009	1.7241
2796	7.1429	10.0000	7.6923	16.6667
2996	2.2222	3.2259	4.1667	6.6667
5005	7.6923	11.1111	8.3333	16.6667

***Table 5**. Models and results of example 2*

Parameters	Neitzel (2010) 'Iterated linearized GHM'	This study: Modified Mixed Model (MMM)	This study: Traditional GHM	This study: Standard GMM
$\hat{c}$	0.9999953579	0.99999662060	0.99999662039	0.99999870015
$\hat{d}$	-0.0000042049*	0.00000488577	0.00000488572	-0.00000083560
$\hat{t}_x$ (m)	29.6432	23.6514	23.6524	16.3939
$\hat{t}_y$ (m)	14.7696	17.3781	17.3780	-9.3872
$\hat{m}$	0.999995357889	0.99999662061	0.99999662040	0.99999870015
$\hat{t}$ (= -t°)	-0.0002409*			
$\hat{t}$ (0≤t<360°)		0.0002799	0.0002799	359.9999521
$\hat{v}^T P \hat{v}$	0.001073 ? (0.000334)	0.001334	0.001334	0.000579
$\hat{\sigma}^2$	0.000179 ? (0.000021)	0.000083	0.000083	0.000097
approx. values from Neitzel	'several' iterations	2 iter.	2 iter.	-
approx. values from GMM	not given	0 iter.	0 iter.	-

***Table 6**. Estimation of errors (m) of Example 2*

Neitzel (2010) / This study				
Point	$\hat{v}_X$	$\hat{v}_Y$	$\hat{v}_x$	$\hat{v}_y$
3	0.0032 0.0040	-0.0025 -0.0026	-0.0055 -0.0068	0.0029 0.0029
185	-0.0066 -0.0074	0.0080 0.0077	0.0066 0.0073	-0.0066 -0.0064
2796	-0.0011 -0.0017	0.0010 0.0012	0.0010 0.0015	-0.0006 -0.0007
2996	-0.0035 -0.0044	0.0070 0.0075	0.0018 0.0024	-0.0034 -0.0036
5005	-0.0014 -0.0015	-0.0007 -0.0010	0.0013 0.0014	0.0005 0.0006

The third example, as shown in Tables 7 to 9, has been taken from Ghilani and Wolf (2006) where their solution obtained by the traditional GHM with initial values given by the GMM. The solution obtained in the first run would be practically the same with that of the next as the authors say. Applying also the traditional

GHM in this study with the same approximate values, the solution is almost identical after two iterations. On the other hand, applying the MMM the solution is slightly different as far as the scale factor concerned. Error estimates are given only as results of MMM and GHM of this study since those from Ghilani and Wolf were not available (expected to be almost the same).

***Table** 7. Observations of different precision*

Example 3: Data taken from Wolf and Ghilani and Wolf (2006)				
Point	Target system (a)		Start system (b)	
	X	Y	x	y
1	-113.000	0.003	0.7637	5.9603
3	0.001	112.993	5.0620	10.5407
5	112.998	0.003	9.6627	6.2430
7	0.001	-112.999	5.3500	1.6540
Point	Standard Deviations			
	σ_X	σ_Y	σ_x	σ_y
1	0.002	0.002	0.026	0.028
3	0.002	0.002	0.024	0.030
5	0.002	0.002	0.028	0.022
7	0.002	0.002	0.024	0.026

***Table 8**. Models and results of example 3*

Parameters	Ghilani and Wolf (2006)	This study		
	Traditional GHM	Modified Mixed Model (MMM)	Traditional GHM	Standard GMM
$\hat{c}$	25.38633347	25.38637009731	25.38633349226	25.38693747693
$\hat{d}$	-0.815897012*	0.81590125888	0.81589701389	0.81460451818
$\hat{t}_x$	-137.2163	-137.2165	-137.2163	-137.2245
$\hat{t}_y$	-150.6000	-150.6002	-150.6000	-150.6039
$\hat{m}$	25.3994412337	25.39947797853	25.39944125601	25.40000344446
$\hat{t}$ (= -t)	-1.8408082*			
$\hat{t}$ ($0 \leq t < 360^o$)		1.8408151	1.8408151	1.8378504
$\hat{v}^T P \hat{v}$	-	0.152017	0.152017	0.072937
$\hat{\sigma}^2$	-	0.012668	0.012668	0.018234
approx. values from GMM	" ≥1 ? "	3 iter.	2 iter.	-

Table 9. *Estimation of errors of example 3*

This study (MMM,GHM)				
Point	$\hat{v}_X$	$\hat{v}_Y$	$\hat{v}_x$	$\hat{v}_y$
1	-0.0000	-0.0000	0.0012	0.0034
3	0.0000	0.0000	-0.0042	-0.0054
5	-0.0000	-0.0000	0.0071	0.0002
7	0.0000	-0.0000	-0.0020	0.0008

Table 10. *Observations of equal precision*

Example 4: Data taken from Sneew et al. (2015)				
Point	Target system (a)		Start system (b)	
	X(m)	Y(m)	x(m)	y(m)
1	19405.518	23159.823	14029.640	12786.840
2	20291.232	22909.817	14914.630	12535.560
3	20150.035	21778.202	14771.830	11404.660
4	18598.550	22211.755	13221.620	11840.320
Point	Weights			
	p_X	p_Y	p_x	p_y
i	1	1	1	1

Table 11. *Models and results of example 4*

Parameters	Sneew et al. (2015)	This study		
	'Mixed model approach-II'	Modified Mixed Model (MMM)	Traditional GHM	Standard GMM
$\hat{c}$	1.00040791931*	1.00040791970	1.00040791927	1.00040791927
$\hat{d}$	-0.00148198750*	-0.00148198793	-0.00148198793	-0.00148198793
$\hat{t}_x$ (m)	5389.091	5389.0913	5389.0913	5389.0913
$\hat{t}_y$ (m)	10347.006	10347.0061	10347.0061	10347.0061
$\hat{m}$	1.000409017	1.00040901739	1.00040901697	1.00040901697
$\hat{t}$ (-90º≤t<90º)	-0.0848769*	(-0.0848769)	(-0.0848769)	(-0.0848769)
$\hat{t}$ (0≤t<360º)	(359.9151230)	359.9151230	359.9151230	359.9151230
$\hat{v}^T P \hat{v}$	0.00128479	0.001285	0.001285	0.002571
$\hat{\sigma}^2$	1.070658E-03*	0.000107	0.000107	0.000643
appr. values from Snew	7 iter.	3 iter.	2 iter.	-
appr. values from GMM	-	0 iter.	0 iter.	-

The fourth example, depicted in Tables 10, 11 and 12, was taken from Sneew et al. (2015), a case with equal data precision. The results among the presented models are almost identical noting only the different number of iterations. Parameters m and t given by Sneew et al. have been converted to c and d for comparison reasons.

Table 12. *Estimation of errors (m) of example 4*

This study(MMM)				
Point	$\hat{v}_X$	$\hat{v}_Y$	$\hat{v}_x$	$\hat{v}_y$
1	0.0068	-0.0154	-0.0068	0.0154
2	0.0021	0.0170	-0.0021	-0.0170
3	-0.0052	-0.0040	0.0052	0.0040
4	-0.0037	-0.0024	0.0037	0.0024

6. Concluding remarks

The Modified Mixed Model (MMM) of the 2D similarity transformation is easily and rigorously applied when all observations in both systems have different precision.

The presented adjustment algorithm is based on the traditional general/mixed model (GHM) and is efficiently simplified in terms of analytical expressions given for the normal equation matrices and for uncorrelated observations. The adjustment process is completed in a number of few iterations.

MMM is not a new idea and has been revisited recently; however, the simplicity of the presented algorithm makes it attractive and efficient for certain applications and software development.

The traditional general model of adjustment gives a solution sufficiently close to that of MMM if the former is properly iterated. In case of equal observation precision MMM and GHM gives the same transformation parameters as those of the GMM. The latter is preferred in many cases when transforming coordinates in a datum with fixed control (common) points in the target system (best fitting process).

The validity of the presented MMM is also proved by its testing in four published examples or experiments.

References

Deakin, R.E. (2007). *Coordinate transformations for Cadastral Surveying*. RMIT University, School of Mathematical and Geospatial Sciences, pp.1-34.

Dermanis A. and A. Fotiou (1992). *Methods and Application of Observation Adjustment (in Greek)*. Ziti Publisher, Thessaloniki.

Fotiou, A. and D. Rossikopoulos (1993). Adjustment, variance component estimation and testing with the affine and similarity transformations. *Zeitschrift für Vermessungswesen,* Heft 10/118, 494-503.

Fotiou A. (2007). *Geometric Geodesy-Theory and Practice*. Ziti editions, Thessaloniki (in Greek).

Fotiou A. and C. Pikridas (2012). *GPS and Geodetic Applications, Second Edition.* Ziti Publications, Thessaloniki, Greece (in greek).

Fotiou A. (2017). Line of Best Fit by a Least Squares Modified Mixed Model. *Living with GIS*, Volume in honour of the memory of Prof. I. Paraschakis, School of Rural and Surveying Engineering, Aristotle University of Thessaloniki, 57-80, Ziti Publishing.

Ghilani C.D. and P.R.Wolf (2006). *Adjustment Computations. Spatial Data Analysis*. John Wiley & Sons, Inc.

Jefferys, H. W. (1980). On the method of least squares. *The Astronomical Journal*, Vol. 85, N0. 2, 177 – 181.

Kaltsikis C.J., I. Paraschakis, D. Rossikopoulos, A. Fotiou (1994). Adjustment algorithms and statistical evaluation of the similarity and affine transformations. *Technika Chronika Sci. Journ. TCG* , Vol. 14, No. 1, 329-351 (in Greek).

Mikhail E.M. and F. Ackermann (1976). *Observations and Least Squares*. IEP-A Dun-Donnelley Publisher.

Neitzel, F. (2010). Generalization of total least-squares on example of unweighted and weighted 2D similarity transformation. J. Geod., 84, 751-762.

Pan G., Y. Zhou, H. Sun H. and W. Guo (2015). Linear observation based total least squares. *Survey Review*, 47:340, 18 – 27.

Paraschakis I. and A. Fotiou (1988). Problems on Coordinate Transformations in map digitization. *Eratosthenes*, No. 21, 125-156 (in Greek).

Schaffrin B. and A. Wieser (2008). On weigthed total least-squares adjustment for linear regression. *J Geod.*, 82, 415– 421.

Simkooei A. A. and S. Jazaeri (2012). Weighted total least squares formulated by standard least squares theory. *Journal of Geodetic Science*, 2(2), 113 – 124.

Sneew, N., F. Krumm, M. Roth (2015). *Adjustment Theory.* Lecture Notes, Rev. 4.43, Geod. Inst. Univ. Stuttgart.