Frame alignment of regional GNSS networks using the Helmert transformation: Old and new perspectives

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Abstract: The mapping problem of an adjusted network from its initial frame to another target frame through the Helmert transformation (HT) is discussed in this paper. We present an optimal solution which can be easily computed by a closed-form expression in terms of appropriate corrections to the standard HT solution that is often used in geodetic practice. Its advantage is the minimization of the propagated noise from the initial network coordinates to their estimated values in the target frame, both at the reference and non-reference stations. This is accomplished by an additional filtering step within the transformation procedure which exploits the known covariance structure of the underlying network in both frames. The presented approach is a suitable tool for aligning an existing network solution to a secular frame such as the ITRF and, as shown in the paper, it can be unequivocally related to the constrained network adjustment directly in the target frame. Nevertheless, any unmodeled non-secular signals hidden in the initial coordinates will be affected by the aforementioned filtering step, and thus the frame alignment methodology presented herein is not tuned for Earth loading studies with respect to a secular reference frame.

1. Introduction

The general prerequisite for the alignment of a geodetic network to ITRF (Altamimi et al. 2011), or to any other global frame of interest, is to process the network measurements along with a subset of reference stations of well known positions with respect to the desired frame. Using the prior information of the reference stations, there are mainly two alternative strategies to express the network coordinates in the desired frame:

- constraining, either stochastically or absolutely, the coordinates of the selected reference stations to their known values during the network adjustment – hereafter called the *constrained network adjustment (CNA) approach*, or
- performing the network adjustment in an unknown or weakly defined frame (e.g. free-net solution) and then fitting the computed solution to the desired frame using a set of Helmert transformation parameters derived from the available reference stations – hereafter called the *Helmert transformation (HT) approach*.

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Both of these approaches have been widely used in practice for geodetic network densification at global, regional and local scale (e.g. Gurtner et al. 1997, Altamimi 2003, Bruyninx et al. 2013). From a theoretical perspective, the first approach is the optimal densification strategy in the sense that the estimated coordinates in the desired frame will have the smallest error variances amongst any other linear unbiased methodology using the same network data (provided that the inverse covariance matrix of the reference coordinates is used as a weight matrix in the constrained adjustment). The HT approach, on the other hand, is able to preserve the network geometry as defined by the actual measurements – something that is not ensured by the CNA approach – through a least-squares fit to the desired frame using the Helmert transformation either in its full form or in an abridged form by omitting some of its original parameters. Its weakness is related to the so-called network configuration effect which can produce biases to the transformed coordinates, especially in regional and local networks (Altamimi 2003). This effect stems from the ill-conditioning in the least-squares adjustment of the HT model over reference stations with limited (non-global) spatial support, and it leads to highly correlated estimates of the transformation parameters and overly reduced accuracy for the transformed coordinates mostly at the non-reference stations.

Following the recent study by Kotsakis et al. (2014), the aim of this paper is to retrace the HT-based approach for network densification and to give a revised formulation which improves, in principle, the accuracy of the estimated coordinates in the target frame. Compared to the usual frame transformation methodology, the presented scheme contains a noise filtering step that reduces the propagated random errors from the initial coordinates to their transformed values by exploiting the known stochastic characteristics of the underlying network. Despite the common availability of the full covariance (CV) matrices of the initial and reference coordinates, this extra step is absent from the determination of Helmert-transformed coordinates in geodetic networks. However, its implementation is easy and it can provide an effective "regularization" tool that may compensate, to some extent, the ill-conditioning (and the resulting instability) caused by the network configuration effect in the final transformation solution. To further support our findings, a useful relationship is derived between the CNA and the HT-based estimators for the station coordinates in the target frame. Besides its theoretical elegance, such a result is particularly useful as not only does it identify the conditions under which the two densification schemes give identical results, but it also justifies the frequently suggested use of an abridged HT model in frame alignment problems.

It is noted that the viewpoint of this study lies on the optimal mapping of an initial network solution to a target frame which is realized by prior coordinates (and their CV matrix) in a subset of the network stations. The aforementioned noise filtering step is an essential part of this procedure, yet it could damp useful hidden information of geodynamical interest within the initial solution that will not be properly transferred for further scientific inference in the target frame (e.g. study of loading displacements). Therefore, the presented approach is a suitable tool for aligning a network solution to a secular frame such as the ITRF, but it will not rightly reproduce non-secular signals in the transformed coordinates. This should be strongly emphasized in order to avoid any confusion to the reader regarding the applicability of our proposed optimal estimator, and it will be further underlined in following sections of the paper.

HT-based network mapping to a target reference frame

The geodetic formulation of the Helmert transformation is commonly expressed as

$$\mathbf{X} = \mathbf{X}' + \mathbf{G}\boldsymbol{\theta} \tag{1}$$

where $\mathbf{X}'$ and $\mathbf{X}$ are the Cartesian coordinate vectors of a set of stations with respect to an initial and a target frame, respectively. The vector $\boldsymbol{\theta}$ contains the frame transformation parameters whereas the matrix $\mathbf{G}$ has a simple form originating from the Jacobian of the nonlinear similarity transformation under sufficiently small orientation and scale differences between the involved frames. The above well known model provides the basis for our following analysis in this section.

At first, let us recall the standard HT approach in network densification problems which is implemented in two steps as follows. Initially, a least-squares estimate of the transformation parameters is obtained from a group of reference stations with known coordinates in both frames. We consider the fully weighted case where the CV matrices of both coordinate sets are used in the estimation process according to the formula:

$$\hat{\boldsymbol{\theta}} = (\mathbf{G}^T (\boldsymbol{\Sigma}_{\mathbf{X}} + \boldsymbol{\Sigma}_{\mathbf{X}'})^{-1} \mathbf{G})^{-1} \mathbf{G}^T (\boldsymbol{\Sigma}_{\mathbf{X}} + \boldsymbol{\Sigma}_{\mathbf{X}'})^{-1} (\mathbf{X} - \mathbf{X}') \tag{2}$$

Subsequently, the estimated Helmert parameters are employed to transform the coordinates of the reference and non-reference stations (denoted by $\mathbf{X}'$ and $\mathbf{Z}'$ respectively) from their initial frame to the target frame:

$$\begin{bmatrix} \hat{\mathbf{x}}^{st} \\ \hat{\mathbf{z}}^{st} \end{bmatrix} = \begin{bmatrix} \mathbf{X}' \\ \mathbf{Z}' \end{bmatrix} + \begin{bmatrix} \mathbf{G} \\ \tilde{\mathbf{G}} \end{bmatrix} \hat{\boldsymbol{\theta}} \tag{3}$$

where the superscript ‘st’ indicates that the computed coordinates are obtained by the standard HT approach. The matrices $\mathbf{G}$ and $\tilde{\mathbf{G}}$ denote the Helmert transformation matrices for the reference and non-reference stations, respectively. Note that the coordinate vectors $\mathbf{X}'$ and $\mathbf{Z}'$ are always correlated with each other as they are jointly obtained by a least-squares network adjustment in some initial frame (which does not need to be further specified at this point).

The solution in Eq. (3) does not arise from a particular estimation principle but it is merely deduced by the forward implementation of the HT model. Although it preserves the network geometry as defined by the adjusted measurements in the initial frame, this solution lacks the property of minimizing the propagated data noise to the transformed network in the target frame. In fact, the random errors of the coordinate vectors $\mathbf{X}'$ and $\mathbf{Z}'$ will be fully absorbed into the result of Eq. (3) which means that the standard approach does not provide full optimal control upon the transformed network coordinates (or, by using "frame terminology", the realization of the target frame in the underlying network through Eq. (3) does not produce station coordinates with optimal accuracy level).

Following a more rigorous approach, the HT-based mapping of a geodetic network to a target frame can be formulated by the combined system of observation equations

$$\mathbf{X} = \mathbf{x} + \mathbf{v_X}\,, \qquad \mathbf{v_X} \sim (\mathbf{0}, \mathbf{\Sigma_X}) \tag{4}$$

$$\mathbf{X}' = \mathbf{x} - \mathbf{G}\boldsymbol{\theta} + \mathbf{v_{X'}}\,, \qquad \mathbf{v_{X'}} \sim (\mathbf{0}, \mathbf{\Sigma_{X'}}) \tag{5}$$

$$\mathbf{Z}' = \mathbf{z} - \tilde{\mathbf{G}}\boldsymbol{\theta} + \mathbf{v_{Z'}}\,, \qquad \mathbf{v_{Z'}} \sim (\mathbf{0}, \mathbf{\Sigma_{Z'}}) \tag{6}$$

in conjunction with the data weight matrix

$$\mathbf{P} = \left[\begin{array}{c|cc} \mathbf{\Sigma_X} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{\Sigma_{X'}} & \mathbf{\Sigma_{X'Z'}} \\ \mathbf{0} & \mathbf{\Sigma_{Z'X'}} & \mathbf{\Sigma_{Z'}} \end{array}\right]^{-1} \tag{7}$$

The latter considers the total statistical information that is available in network densification problems, whereas the vectors $\mathbf{x}$ and $\mathbf{z}$ correspond to the true coordinates of the reference and non-reference stations in the target frame. The analytic form of the weighted least-squares solution of the previous system is given in Kotsakis et al. (2014). Therein it was shown that the adjusted Helmert parameters remain the same as in Eq. (2), a fact that is expected since the inclusion of the non-reference stations into the adjustment procedure does not contribute any additional information for the determination of those parameters. On the other hand, the derived least-squares solution of the network coordinates in the target frame differs from the classic expression in Eq. (3) as follows (*ibid.*)

$$\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}}^{st} \\ \hat{\mathbf{z}}^{st} \end{bmatrix} + \begin{bmatrix} \mathbf{\Sigma_{X'}} \\ \mathbf{\Sigma_{Z'X'}} \end{bmatrix} (\mathbf{\Sigma_X} + \mathbf{\Sigma_{X'}})^{-1} (\mathbf{X} - \hat{\mathbf{x}}^{st}) \tag{8}$$

The above formula gives the optimal HT-based estimator in terms of additive corrections to the standard HT-based estimator for network densification purposes. Both of these estimators produce a network solution that refers to the same target frame, namely to the one realized by the prior coordinates $\mathbf{X}$ of the reference stations. Their difference is that Eq. (8) leads to station coordinates with smaller error variances compared to the standard estimates $\hat{\mathbf{x}}^{\mathrm{st}}$ and $\hat{\mathbf{z}}^{\mathrm{st}}$, as it has been analytically shown in Kotsakis et al. (2014).

Evidently, Eq. (3) and Eq. (8) give identical results under the conditions $\mathbf{\Sigma}_{\mathbf{X}'} = \mathbf{0}$ and $\mathbf{\Sigma}_{\mathbf{Z}'\mathbf{X}'} = \mathbf{0}$, none of which ever applies in practice, at least for cases of adjusted networks that need to be transformed to another frame. Furthermore, if the prior coordinates of the reference stations are assumed errorless ($\mathbf{\Sigma}_{\mathbf{X}} = \mathbf{0}$) then the optimal HT solution in Eq. (8) will reproduce their values ($\hat{\mathbf{x}} = \mathbf{X}$) similarly to the rationale of the constrained network adjustment directly in the desired frame. In general, though, the CNA and HT densification schemes do not give the same estimated coordinates at the non-reference stations. This is because each of these schemes defines the target frame at a different stage during the network analysis, that is, either *in tandem* with the adjustment of the network measurements (CNA approach) or *after* the adjustment of the network measurements in an arbitrary initial frame (HT approach). Their actual differences are discussed and evaluated in more detail in the following section.

As a final note, let us stress that the inherent filtering in Eq. (8) suppresses the noise of the initial solution ($\mathbf{X}'$, $\mathbf{Z}'$) provided of course that all relevant CV matrices are correct or, at least, realistic. This noise-filtering step is missing from the standard HT solution in Eq. (3), thus making the transformed coordinates ($\hat{\mathbf{x}}^{\mathrm{st}}$, $\hat{\mathbf{z}}^{\mathrm{st}}$) to have larger error variances compared to the result of Eq. (8). On the other hand, in frame alignment applications for generating coordinate time series towards geodynamical investigations (e.g. Tregoning and van Dam 2005, Bevis and Brown 2014) such a filtering step may not be desirable as it could weaken the signal information hidden in the initial solution. This is especially true when $\mathbf{\Sigma}_{\mathbf{X}} \ll \mathbf{\Sigma}_{\mathbf{X}'}$, in which case the transformed network will be forced to "follow" the secular character of the target frame (e.g. ITRF) thus obscuring any non-secular geophysical signals originating by unmodeled loading effects. However, the importance of Eq. (8) remains in the sense of an effective tool for the combination of independent overlapping networks using their full covariance information in their respective frames – see also Kotsakis et al. (2014) where the more general case of inter-correlated overlapping networks is treated.

Comparison of the CNA and HT-based estimators in network densification

For the purpose of this study, it is instructive to relate the CNA and HT densification strategies when using the same set of reference stations. To compare them in an analytic way we consider the normal equations (NEQ) originating from the data processing in the underlying network (before adding any datum constraints and after eliminating any auxiliary parameters from the adjustment procedure)

$$\mathbf{N}\begin{bmatrix}\mathbf{x}-\mathbf{x}_o\\ \mathbf{z}-\mathbf{z}_o\end{bmatrix} = \mathbf{u} \tag{9}$$

where $\mathbf{x}_o$ and $\mathbf{z}_o$ are the approximate coordinates for the reference and non-reference stations that are used in the linearization of the NEQ system.

Restricting our attention to GNSS networks, the above system is generally invertible as it already contains the auxiliary datum information from the adopted IGS orbits. An initial "free" network solution can therefore be obtained as

$$\begin{bmatrix}\mathbf{X}'\\ \mathbf{Z}'\end{bmatrix} = \begin{bmatrix}\mathbf{x}_o\\ \mathbf{z}_o\end{bmatrix} + \mathbf{N}^{-1}\mathbf{u} \tag{10}$$

which is given in a reference frame that could be far from the ITRF since it is realized only at the precision level of the IGS orbits (a few cm). This solution can be transformed to any desired frame, e.g. ITRF, using the HT-based scheme that was described in the previous section.

Alternatively, a constrained solution directly in the desired frame can be obtained via the same prior information for the reference stations ($\mathbf{X}$, $\mathbf{\Sigma_X}$) according to the well known least-squares adjustment formula

$$\begin{bmatrix}\hat{\mathbf{x}}^c\\ \hat{\mathbf{z}}^c\end{bmatrix} = \begin{bmatrix}\mathbf{x}_o\\ \mathbf{z}_o\end{bmatrix} + (\mathbf{N}+\mathbf{H}^T\mathbf{\Sigma}_\mathbf{X}^{-1}\mathbf{H})^{-1}(\mathbf{u}+\mathbf{H}^T\mathbf{\Sigma}_\mathbf{X}^{-1}(\mathbf{X}-\mathbf{x}_o)) \tag{11}$$

where the auxiliary matrix $\mathbf{H}$ has the partitioned form $[\mathbf{I} \mid \mathbf{0}]$ in accordance to the NEQ partitioning in Eq. (9). In general, the above solution differs from the HT solution of Eq. (8) according to the formula (see proof in the next section)

$$\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{z}}\end{bmatrix} - \begin{bmatrix}\hat{\mathbf{x}}^c\\ \hat{\mathbf{z}}^c\end{bmatrix} = (\mathbf{N}+\mathbf{H}^T\mathbf{\Sigma}_\mathbf{X}^{-1}\mathbf{H})^{-1}\mathbf{N}\mathbf{E}^T\hat{\boldsymbol{\theta}} \tag{12}$$

where $\hat{\boldsymbol{\theta}}$ is given by Eq. (2) and $\mathbf{E}$ is the transformation matrix for the entire network (reference + non-reference stations)

$$\mathbf{E} = \begin{bmatrix}\mathbf{G}^T & \tilde{\mathbf{G}}^T\end{bmatrix} \tag{13}$$

It is seen that the difference of the two solutions depends on the frame transformation model that is employed in the HT approach. This is not a trivial realization and, actually, it can explain the fact that the full (7-parameter) Helmert model might not always be the best choice to obtain a well-expressed GNSS network solution in the desired frame. In fact, Eq. (12) shows that the weighted CNA and HT-based estimators give the same result under the condition $\mathbf{N}\mathbf{E}^T = \mathbf{0}$ which implies that the selected transformation model should contain only the parameters that correspond to the datum defect of the underlying network (e.g. Blaha 1971, Sillard and Boucher 2001).

In the case of GNSS networks the NEQ system in Eq. (9) is generally invertible ($\mathbf{N}\mathbf{E}^T \neq \mathbf{0}$), thus causing an unavoidable offset between the constrained solution of Eq. (11) and the transformed solution of Eq. (8), at least at the non-reference stations, for any choice of the transformation model. However, if an abridged model is employed in the HT approach, e.g. shift-only model, then the difference of the two solutions will be dictated by (a linear combination of) the columns of the matrix $\mathbf{N}\mathbf{E}^T$ that correspond only to the selected model parameters. This means that their consistency may be improved if the omitted parameters correspond to numerically large columns of the aforementioned matrix. Typically, such frame parameters in GNSS networks are the rotation angles and the scale factor, which could cause apparent biases to the transformed coordinates in the desired frame relative to the weighted CNA solution.

To provide a quick example of the differences among the network densification schemes, we have used daily sinex files from regional subnetworks that are regularly processed by local analysis centers (LACs) of the European Permanent Net-

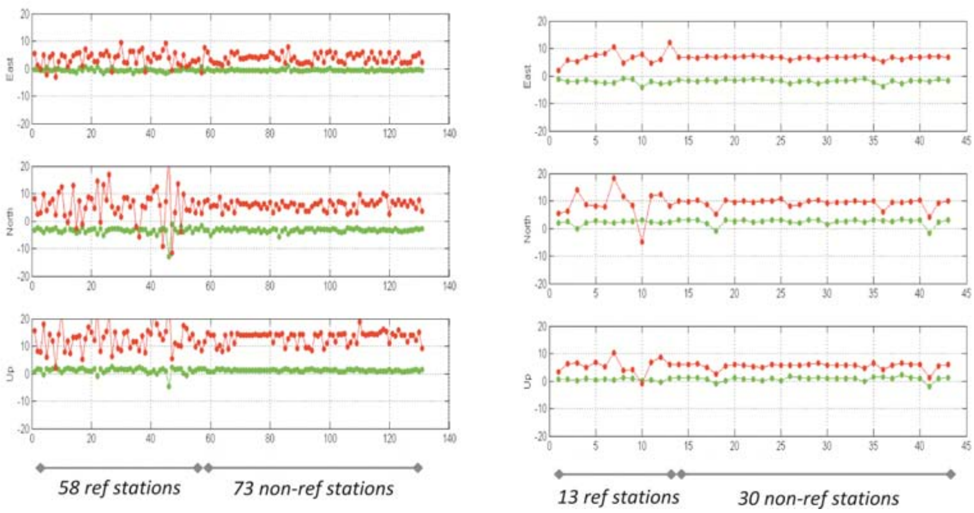

Fig. 1 *Differences (in mm) of the ITRF2008 coordinates obtained by the weighted CNA solution vs. the optimal HT solution (green line) and the standard HT solution (red line). The used sinex files refer to the sixth day of GPS week 1809 from the MUT and SUT (left and right plots respectively) local analysis centers of the EPN network.*

work (EPN). Using the unconstrained NEQs of each subnetwork, we computed and compared the ITRF2008 coordinates obtained by Eq. (3) (*standard HT approach*), Eq. (8) (*optimal HT approach*) and Eq. (11) (*weighted CNA approach*) based on the same reference stations in each case. The prior coordinates of the reference stations and their full CV matrix were extracted from the ITRF2008-TRF-IGS sinex file (itrf.ensg.ign.fr/ITRF_solutions/2008/ITRF2008_files.php) and were reduced, by their known velocities, to the current epoch of the daily solutions. Some representative results from two different subnetworks are given in Fig. 1. The apparent biases in the standard HT solution due to the network configuration effect are clearly visible, and they amount to several mm in both horizontal and vertical components. The optimal HT solution, on the other hand, seems to provide a much better agreement with the weighted CNA solution over all stations in every case.

The analytic proof of Eq. (12)

Starting from Eq. (8), and using the auxiliary matrix $\mathbf{H} = [\mathbf{I} \mid \mathbf{0}]$, the optimal HT solution can be equivalently expressed as

$$\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}}^{\mathrm{st}} \\ \hat{\mathbf{z}}^{\mathrm{st}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{X}'} & \boldsymbol{\Sigma}_{\mathbf{X}'\mathbf{Z}'} \\ \boldsymbol{\Sigma}_{\mathbf{Z}'\mathbf{X}'} & \boldsymbol{\Sigma}_{\mathbf{Z}'} \end{bmatrix} \mathbf{H}^T \left(\boldsymbol{\Sigma}_{\mathbf{X}} + \mathbf{H} \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{X}'} & \boldsymbol{\Sigma}_{\mathbf{X}'\mathbf{Z}'} \\ \boldsymbol{\Sigma}_{\mathbf{Z}'\mathbf{X}'} & \boldsymbol{\Sigma}_{\mathbf{Z}'} \end{bmatrix} \mathbf{H}^T \right)^{-1} (\mathbf{X} - \hat{\mathbf{x}}^{\mathrm{st}}) \qquad (14)$$

Taking into account Eq. (3) and also the matrix notation from Eq. (13), the previous equation becomes

$$\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \mathbf{X}' \\ \mathbf{Z}' \end{bmatrix} + \mathbf{E}^T \hat{\boldsymbol{\theta}} + \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{X}'} & \boldsymbol{\Sigma}_{\mathbf{X}'\mathbf{Z}'} \\ \boldsymbol{\Sigma}_{\mathbf{Z}'\mathbf{X}'} & \boldsymbol{\Sigma}_{\mathbf{Z}'} \end{bmatrix} \mathbf{H}^T \left(\boldsymbol{\Sigma}_{\mathbf{X}} + \mathbf{H} \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{X}'} & \boldsymbol{\Sigma}_{\mathbf{X}'\mathbf{Z}'} \\ \boldsymbol{\Sigma}_{\mathbf{Z}'\mathbf{X}'} & \boldsymbol{\Sigma}_{\mathbf{Z}'} \end{bmatrix} \mathbf{H}^T \right)^{-1} \times$$
$$\times (\mathbf{X} - \mathbf{X}' - \mathbf{H}\mathbf{E}^T \hat{\boldsymbol{\theta}}) \qquad (15)$$

For the sake of convenience we introduce the auxiliary symbols:

$$\hat{\boldsymbol{\xi}} = \begin{bmatrix} \hat{\mathbf{x}} - \mathbf{x}_{\mathrm{o}} \\ \hat{\mathbf{z}} - \mathbf{z}_{\mathrm{o}} \end{bmatrix}, \quad \boldsymbol{\xi}' = \begin{bmatrix} \mathbf{X}' - \mathbf{x}_{\mathrm{o}} \\ \mathbf{Z}' - \mathbf{z}_{\mathrm{o}} \end{bmatrix}, \quad \boldsymbol{\Sigma}_{\boldsymbol{\xi}'} = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{X}'} & \boldsymbol{\Sigma}_{\mathbf{X}'\mathbf{Z}'} \\ \boldsymbol{\Sigma}_{\mathbf{Z}'\mathbf{X}'} & \boldsymbol{\Sigma}_{\mathbf{Z}'} \end{bmatrix} \qquad (16)$$

and thus Eq. (15) takes the form

$$\hat{\boldsymbol{\xi}} = \boldsymbol{\xi}' + \mathbf{E}^T \hat{\boldsymbol{\theta}} + \boldsymbol{\Sigma}_{\boldsymbol{\xi}'} \mathbf{H}^T \left(\boldsymbol{\Sigma}_{\mathbf{X}} + \mathbf{H} \boldsymbol{\Sigma}_{\boldsymbol{\xi}'} \mathbf{H}^T \right)^{-1} (\mathbf{X} - \mathbf{H}\boldsymbol{\xi}' - \mathbf{x}_{\mathrm{o}} - \mathbf{H}\mathbf{E}^T \hat{\boldsymbol{\theta}}) \qquad (17)$$

Using the well known matrix identity (Schaffrin 1983, p. 34, Eq. (A12))

$$\mathbf{D}\mathbf{C}(\mathbf{A} + \mathbf{B}\mathbf{D}\mathbf{C})^{-1} = (\mathbf{I} + \mathbf{D}\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{D}\mathbf{C}\mathbf{A}^{-1} \qquad (18)$$

we have

$$\boldsymbol{\Sigma}_{\xi'}\mathbf{H}^T\left(\boldsymbol{\Sigma}_\mathbf{X}+\mathbf{H}\boldsymbol{\Sigma}_{\xi'}\mathbf{H}^T\right)^{-1} = \left(\mathbf{I}+\boldsymbol{\Sigma}_{\xi'}\mathbf{H}^T\boldsymbol{\Sigma}_\mathbf{X}^{-1}\mathbf{H}\right)^{-1}\boldsymbol{\Sigma}_{\xi'}\mathbf{H}^T\boldsymbol{\Sigma}_\mathbf{X}^{-1} \tag{19}$$

which allows us to express Eq. (17) as

$$\hat{\boldsymbol{\xi}} = \boldsymbol{\xi}' + \mathbf{E}^T\hat{\boldsymbol{\theta}} + \left(\mathbf{I}+\boldsymbol{\Sigma}_{\xi'}\mathbf{H}^T\boldsymbol{\Sigma}_\mathbf{X}^{-1}\mathbf{H}\right)^{-1}\boldsymbol{\Sigma}_{\xi'}\mathbf{H}^T\boldsymbol{\Sigma}_\mathbf{X}^{-1}(\mathbf{X}-\mathbf{H}\boldsymbol{\xi}'-\mathbf{x}_\mathrm{o}-\mathbf{H}\mathbf{E}^T\hat{\boldsymbol{\theta}}) \tag{20}$$

Multiplying both sides of the last equation by the matrix $\mathbf{I}+\boldsymbol{\Sigma}_{\xi'}\mathbf{H}^T\boldsymbol{\Sigma}_\mathbf{X}^{-1}\mathbf{H}$ and after some simple algebraic manipulations, we get

$$\left(\mathbf{I}+\boldsymbol{\Sigma}_{\xi'}\mathbf{H}^T\boldsymbol{\Sigma}_\mathbf{X}^{-1}\mathbf{H}\right)\hat{\boldsymbol{\xi}} = \boldsymbol{\xi}' + \mathbf{E}^T\hat{\boldsymbol{\theta}} + \boldsymbol{\Sigma}_{\xi'}\mathbf{H}^T\boldsymbol{\Sigma}_\mathbf{X}^{-1}(\mathbf{X}-\mathbf{x}_\mathrm{o}) \tag{21}$$

Let us now introduce the auxiliary vector

$$\hat{\boldsymbol{\xi}}^c = \begin{bmatrix}\hat{\mathbf{x}}^c - \mathbf{x}_\mathrm{o} \\ \hat{\mathbf{z}}^c - \mathbf{z}_\mathrm{o}\end{bmatrix} \tag{22}$$

which corresponds to the constrained solution directly in the desired frame using the same reference stations with the HT solution. Based on Eq. (11) we obviously have

$$\mathbf{H}^T\boldsymbol{\Sigma}_\mathbf{X}^{-1}(\mathbf{X}-\mathbf{x}_\mathrm{o}) = (\mathbf{N}+\mathbf{H}^T\boldsymbol{\Sigma}_\mathbf{X}^{-1}\mathbf{H})\hat{\boldsymbol{\xi}}^c - \mathbf{u} \tag{23}$$

By substituting the last expression into Eq. (21) and after performing straightforward operations, we get

$$\left(\mathbf{I}+\boldsymbol{\Sigma}_{\xi'}\mathbf{H}^T\boldsymbol{\Sigma}_\mathbf{X}^{-1}\mathbf{H}\right)\hat{\boldsymbol{\xi}} = \boldsymbol{\xi}' + \mathbf{E}^T\hat{\boldsymbol{\theta}} + \left(\mathbf{I}+\boldsymbol{\Sigma}_{\xi'}\mathbf{H}^T\boldsymbol{\Sigma}_\mathbf{X}^{-1}\mathbf{H}\right)\hat{\boldsymbol{\xi}}^c + \left(\boldsymbol{\Sigma}_{\xi'}\mathbf{N}-\mathbf{I}\right)\hat{\boldsymbol{\xi}}^c - \boldsymbol{\Sigma}_{\xi'}\mathbf{u} \tag{24}$$

Considering that the network solution $\boldsymbol{\xi}'$ in the initial frame is obtained by a free-net adjustment (see Eq. (10)) we have $\mathbf{N}\boldsymbol{\xi}'=\mathbf{u}$ and $\boldsymbol{\Sigma}_{\xi'}=\mathbf{N}^{-1}$, and thus the last equation yields

$$\left(\mathbf{I}+\boldsymbol{\Sigma}_{\xi'}\mathbf{H}^T\boldsymbol{\Sigma}_\mathbf{X}^{-1}\mathbf{H}\right)\hat{\boldsymbol{\xi}} = \mathbf{E}^T\hat{\boldsymbol{\theta}} + \left(\mathbf{I}+\boldsymbol{\Sigma}_{\xi'}\mathbf{H}^T\boldsymbol{\Sigma}_\mathbf{X}^{-1}\mathbf{H}\right)\hat{\boldsymbol{\xi}}^c \tag{25}$$

or equivalently

$$\begin{aligned}\hat{\boldsymbol{\xi}} &= \hat{\boldsymbol{\xi}}^c + \left(\mathbf{I}+\boldsymbol{\Sigma}_{\xi'}\mathbf{H}^T\boldsymbol{\Sigma}_\mathbf{X}^{-1}\mathbf{H}\right)^{-1}\mathbf{E}^T\hat{\boldsymbol{\theta}} \\ &= \hat{\boldsymbol{\xi}}^c + \left(\mathbf{N}^{-1}\mathbf{N}+\mathbf{N}^{-1}\mathbf{H}^T\boldsymbol{\Sigma}_\mathbf{X}^{-1}\mathbf{H}\right)^{-1}\mathbf{E}^T\hat{\boldsymbol{\theta}} \\ &= \hat{\boldsymbol{\xi}}^c + \left(\mathbf{N}+\mathbf{H}^T\boldsymbol{\Sigma}_\mathbf{X}^{-1}\mathbf{H}\right)^{-1}\mathbf{N}\mathbf{E}^T\hat{\boldsymbol{\theta}}\end{aligned} \tag{26}$$

Taking into account Eqs. (16) and (22) we finally have

$$\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}}^c \\ \hat{\mathbf{z}}^c \end{bmatrix} + \left(\mathbf{N}+\mathbf{H}^T \boldsymbol{\Sigma}_{\mathbf{X}}^{-1} \mathbf{H}\right)^{-1} \mathbf{N}\mathbf{E}^T \hat{\boldsymbol{\theta}} \quad (27)$$

which concludes our proof.

Conclusions

The mapping problem of a network solution to a target frame through the HT approach was discussed in this paper. It has been shown that the optimal coordinates in the target frame (in the sense of minimum estimation error variance) can be computed by a closed-form expression in terms of appropriate corrections to the standard HT-based estimator that is commonly used in geodetic practice. Our revised estimator is easy to implement and it does not require any additional matrix inversion other than the one already used by the classic stepwise solution of Eqs. (2) and (3). Also, it was proved that its difference with the weighted CNA estimator depends, expectedly, on the chosen transformation model and, in particular, on the linear combination of the columns of the matrix $\mathbf{N}\mathbf{E}^T$; see Eq. (12). This is a useful result as it implies the equivalency of the two general approaches for network densification when the frame transformation model ($\boldsymbol{\theta}$, $\mathbf{E}$) employs only the parameters related to the rank defect of the underlying network.

The advantage of the optimal HT approach presented here, compared to the standard HT approach, is the minimization of the propagated noise from the initial network solution to the estimated coordinates in the target frame (both at the reference and non-reference stations). This is accomplished by a filtering step within the transformation procedure which exploits the network's known covariance structure in both frames; see Eq. (8). It is again noted that any unmodeled non-secular signals in the initial coordinates will be affected by such filtering during their transfer to the target frame. Hence, the methodology described in this paper is not tuned for the analysis of Earth loading signals with respect to a secular reference frame, although the aforementioned noise reduction is a critical and worth-considering aspect in support of geophysical signal detection in ITRF-aligned coordinate time series.

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