Reduced point mass or multipole base functions

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Abstract

Point-mass functions or multipole base-functions are harmonic functions, which may be used to represent the (anomalous) gravity potential (*T*) globally or locally. The functions may be expressed by closed expressions or as sums of Legendre series. In both cases at least the two first terms must be removed since they are not present in *T*. For local applications the effect of a global gravity model is generally removed (and later restored). Then more terms need to be removed or substituted by terms similar to error-degree variances. We have done some calculations to illustrate the effect of reducing the point mass or multipole functions, i.e. showing how the first zero-crossing as a function of spherical distance comes closer to zero when more terms are removed.

1. Introduction

Linear combinations of point mass functions or mass multipoles have been used for the representation of the global (***W***) or regional anomalous gravity potential, ***T***, see e.g. Balmino (1974), Hauck and Lelgemann (1985), Vermeer (1982, 1989, 1990,1992,1993), Marchenko et al., (2001), Ballani et al. (1993), Wu (1984).

The anomalous potential *T*, is equal to the difference between *W* and a global gravity field model like EGM96 (Lemoine et al., 1998), ie. it is a harmonic function.

For a point mass base function we have for an approximation to *T*:

$$\tilde{T} = \sum_{i=1}^{I} \frac{GM_i}{l_i} \qquad (1)$$

where G is the gravitational constant, M_i is the mass, I the number of point masses and l_i is the distance from the mass located at the point Q_i to the point of evaluation, P, see Fig. 1.

The distance from the origin to P and Q_i is denoted r_P , r_{Q_i} , respectively and the first will always be larger than the other. The angle (spherical distance) between the vectors to P and Q_i is denoted ψ.
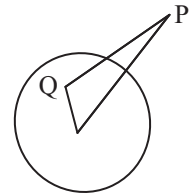

Fig 1

We now drop the subscript on Q. The distance l, from P to Q is then $l = \sqrt{r_P^2 + r_Q^2 - 2r_P r_Q \cos(\psi)}$. For the inverse distance we have

$$\frac{1}{l} = \frac{1}{r_P} \sum_{i=0}^{\infty} \left(\frac{r_Q}{r_P}\right)^i P_i(\cos\psi) \tag{2}$$

where P_i are the Legendre polynomials. Multipole-functions are derive from the inverse distance function by integration or differentiation (compare Tscherning and Rapp (1974)), and will be denoted f.

$$\tilde{T}(P) = \sum_{i=1}^{M} a_i f_i(P) \tag{3}$$

The estimate $\tilde{T}$ is determined so that $\sum_{i=1}^{M} \left(\{L_i \tilde{T}\} - obs_i\right)^2 = \min$, where M is the number of observations and L_i are the linear functionals associated with the observations. In the computations described below we only consider height anomalies (geoid heights), gravity disturbances and second-order radial derivatives.

We intend to show that in order to make the functions suitable for regional gravity field modeling low-degree terms may be removed or substituted by appropriate weights.

2. Higher order derivatives

For derivatives with respect to r_P we have series expansions similar to eq. (2), where the terms for the first derivative are multiplied with $-(i+1)/r_P$ and for the second derivative with $(i+1) \times (i+2)/r_P^2$. Closed expressions for the derivatives are easily found.

$$\frac{\partial \frac{1}{l}}{\partial r_P} = \frac{(r_Q t - r_P)}{l^3} \tag{4}$$

$$\frac{\partial^2 \frac{1}{l}}{\partial r_P^2} = \frac{\frac{3(r_P - r_Q t)^2}{l^2} - 1}{l^3} \tag{5}$$

Point masses are not the only harmonic functions which may be used as base functions when approximating T, see e.g. Tscherning (1972), Hauck and Lelgemann (1985) but we will only deal with point mass and excentric multipoles base functions, since they fully represents the message of this paper.

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3. Reduced point masses

From eq. (2) we see that if we use the representation eq. (1), it will contain terms of degree zero and one, which are not present in T. So they have to be removed, simply by subtracting from the closed expressions the first two terms in eq. (2) or its derivative. (This is not done for the examples of closed expressions in section 5). But what if we subtract from the data (and later add back) the contribution from a global model like EGM96, complete to degree N ?

$$\bar{T}_N - \frac{GM}{r_P} \sum_{n=0}^{N} \left(\frac{a}{r_P}\right)^n \sum_{m=-n}^{n} C_{nm} P_{nm}(\sin\varphi) \begin{Bmatrix} \cos(m\lambda), & m \geq 0 \\ \sin(|m|\lambda), & m < 0 \end{Bmatrix} \tag{6}$$

Here φ is the geocentric latitude, λ the longitude, P_{nm} the normalized Legendre functions and a a scale factor close to the semi-major axis. C_{nm} are the normalized Stokes coefficients with error-estimates σ_{nm}^2. Degree-variances and error-degree-variances are sums of the C_{nm} squared, σ_{nm}^2. respectively, for a fixed degree, n, multiplied with $(GM/a)^2$.

As pointed out by Arabelos (1980) we can not simply put to zero the first N terms. Here a solution was found, i.e. that the first terms were not put to zero, but put equal to the so-called error-degree variances, $\sigma_{e,i}^2$ contingently scaled by a factor α so as to reflect if the model was better ($\alpha \leq 1$) or worse ($\alpha \geq 1$) in an area.

To get a little more insight into this, let us interpret the point mass potential as a reproducing kernel in a Hilbert space, where the functions are harmonic down to a Bjerhammar-sphere with radius R_B inside the Earth and $r_Q < R_B$. Using a Kelvin transformation we obtain a point D outside the sphere

$$\frac{R_B^2}{r_P r_D} = \frac{r_Q}{r_P} \tag{7}$$

So that $L = \sqrt{1 - 2\frac{R_B^2}{r_P r_D}\cos(\psi) + \left(\frac{R_B^2}{r_P r_D}\right)^2}$. Then $K(P,D) = \dfrac{\dfrac{r_B^2}{r_P r_D}}{L}$ is the so-called

Krarup kernel, Krarup (1969). When interpreted as a covariance function, it has unitless degree-variances equal to 1, i.e. it is not well suited to represent the anomalous potential, since the degree-variances of T tends to zero like $n^{-3} * q$, $q < 1$, see Tscherning and Rapp (1974). So instead of point mass base functions one should consider using potentials of other types of masses, like a bar, see Hauck and Lelgemann (1985, Fig. 2) or excentric multipoles (Marchenko et al., 2001).

There are other inherent problems using mass-type base functions. Which depth should be used for the masses. Should they form a grid ? (Vermeer (1990)). This has been studied extensively by e.g. Barthelmes and Kautzleben (1983) and

Barthelmes (1985). We will not discuss this here, but show tables of reduced point mass functions.

4. Excentric multipoles – covariance functions

The covariance functions generally used in least-squares collocation may be interpreted as excentric multipoles using the Kelvin transformation. For two points outside the Earth, P and D, we have typically (Tscherning and Rapp, (1974)) for the covariance of the anomalous potential,

$$\text{cov}(r_P, r_D, \psi) = \sum_{i=2}^{\infty} \frac{A}{(i-1)(i-2)(i+4)} \left(\frac{R_B^2}{r_D r_P} \right)^{i+1} P_i(\cos\psi) \tag{8}$$

which is the linear combination of 3 excentric multipoles:

$$F_k(r_P, r_Q, \psi) = \sum_{i=2}^{\infty} \frac{A_k}{(i-k)} \left(\frac{r_Q}{r_P} \right)^{i+1} P_i(\cos\psi) \tag{9}$$

Each of these may be represented by a closed expression. The first terms corresponding to the maximal degree of the global model used, should be substituted by the (scaled potential) error-degree-variances.

$$\text{cov}_N(r_P, r_D, \psi) = \sum_{i=2}^{\infty} \alpha \sigma_i^e \left(\frac{R_B^2}{r_D r_P} \right)^{i+1} P_i(\cos\psi) \tag{10}$$

5. Reduced point mass and excentric multipole base functions – examples

The interesting functions are those which represent the geoid (height anomaly) and the first and second derivative with respect to r_P. All quantities are anomalous quantities with respect to EGM96 complete to degree N=24. We consider geoid and gravity disturbance at height zero and the second order radial derivative at 250 km altitude. For the latter we will put the mass-point at depth 242.5 km, corresponding to data (r_P) at 250 km. The Bjerhammar sphere will be put at 1.46 km depth.

In the first example (Table 1) we use as observation **one** (anomalous) radial gravity gradient observation equal to 0.15 Eötvöes, at height 250 km ($M = 1$ in eq. (3)). The latitude and the longitude are set to zero for the mass-point, but the latitude for the point P increases in steps from zero. Table 2 shows the same function, now with the quantities evaluated at zero altitude.

Table 1: *Closed and Reduced Point mass functions at altitude 250 km. Note the location of the first zero-crossing. For the reduced function all terms up to degree 24 have been set equal to zero.*

ψ	Closed expression Geoid (m)	Reduced to deg. 24	Closed Gravity disturbance (mgal)	Reduced	Closed 2. order radial deriv. (Eötvös)	Reduced
0.0	0.85	0.29	−2.41	−1.79	0.150	0.150
1.0	0.80	0.24	−2.03	−1.37	0.106	0.100
2.0	0.70	0.14	−1.34	−0.62	0.043	0.028
3.0	0.59	0.05	−0.81	−0.09	0.012	−0.005
4.0	0.49	−0.02	−0.49	0.16	0.002	−0.014
5.0	0.42	−0.05	−0.31	0.24	−0.001	0.013
6.0	0.37	−0.05	−0.21	0.23	−0.002	−0.010
7.0	0.32	−0.05	−0.14	0.17	−0.002	−0.007
8.0	0.29	−0.03	−0.11	0.10	−0.001	−0.003
9.0	0.26	−0.01	−0.10	0.01	−0.001	0.000
10.0	0.23	0.00	−0.06	−0.02	0.000	0.002

Table 2: *Closed and Reduced Point mass functions at zero altitude. Note the location of the first zero-crossing and the values for zero spherical distance. The location is for the geoid at 3°. This corresponds (approximately) to the location of the first zero-point for the first Legendre polynomium in the series eq. (8). (Legendre polynomials have as many zero points as the degree in the interval from -90° to 90° distributed approximately equidistantly, see Heiskanen and Moritz, 1967, Fig. 1-8).*

ψ	Closed expression Geoid (m)	Reduced to deg. 24	Closed Gravity disturbance (mgal)	Reduced	Closed 2. order radial deriv. (Eötvös)	Reduced
0.0	3.56	3.13	−49.2	−55.3	13.9	16.1
1.0	1.93	1.24	−7.87	−7.26	−0.11	−0.18
2.0	1.09	0.31	−1.48	0.04	−0.14	−0.21
3.0	0.75	0.02	−0.51	0.98	−0.053	−0.010

Table 3: *Values computed using a degree-variance model (eq.(8)) with depth to the Bjerhammar-sphere equal to 1.56 km. Error.degree-variances from EGM96 used to degree 24, scaled with α = 1.03. Computed using Least-Squares-Collocation with only one observation (completely equivalent to using eq. (3)).*

ψ	Second order radial derivative. Altitude= 250 km. Eöetvöes units.	Geoid. Zero altitude. Units m.	Gravity anomaly. Zero altitude. Units mgal.
0.0	0.15	2.11	13.11
1.0	0.13	1.80	10.36
2.0	0.08	1.23	5.33
3.0	0.03	0.46	1.40
4.0	−0.03	−0.04	−0.87
5.0	−0.03	−0.32	−1.86
6.0	−0.03	−0.43	−2.01
7.0	−0.02	−0.41	−1.66
8.0	−0.02	−0.30	−1.08

In Table 3 we have corresponding values obtained using the covariance function eq. (8), (10) and again with the same gravity gradient observation as used in Table 1 and 2.

We have here calculated values with spherical distance 1.0 degree and so that the value for the second order radial derivative is 0.15 Eötvös.

Again we see the location of the first zero-crossing, which makes the function well suited for representing data where the contribution from EGM96 to degree 24 has been subtracted.

A FORTRAN program redpmass.f for doing these or similar calculations is available as http://cct.gfy.ku.dk/redpmass.f , 2010.05.20.

4. Conclusion

Closed or reduced point mass or multipole functions may be used to represent the anomalous potential. When used regionally referring to a global gravity field model, the first terms must be removed or substituted by error-degree-variances.

For point-mass or multipole functions the terms up to the lowest degree of the reference potential (the global model) have here been put equal to zero. However, it might be possible to find (unitless) terms representing the power in the frequencies which the global model have not removed, corresponding to error-degree variances.

If covariance functions (corresponding to multipole base functions) are used, error degree-variances (scaled) may be used. This assures that the model in an appropriate manner weights the regional frequencies with respect to the global model used.

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