Analytical expressions for the gravitational potential and attraction of a right circular cone for points situated along its symmetry axis

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Abstract

Basic tool in the process of gravity field modeling and interpretation is the forward computation of the gravitational effect of mass distributions of known geometry and consistency. Apart from the standard works of Jung (1961) and Talwani (1973), where closed expressions for the gravity anomaly of a conical mass distribution for a point situated at special locations at its symmetry axis can be found, the case study of a right rectangular cone has not received the attention of other ideal bodies in the relevant geodetic or geophysical literature, such as the right rectangular prism, cylinder or general polyhedron. Scope of the present contribution is to present a thorough overview of the mathematical nomenclature linked to the problem of expressing the gravitational effect of a conical mass distribution of constant density in terms of a closed analytical solution. Closed analytical expressions are presented for the potential and its vertical derivative for random locations of the computation point at the symmetry axis. Thereby, a basic distinction concerning the height-to-radius ratio of the attracting conical source is performed.

1. Introduction

Many independent tools and algorithms exist for the purpose of interpreting potential field data. These methodologies can be sorted into two basic categories, forward and inverse methods. However, some procedures are common to these two areas as they serve in certain steps both the forward and the inverse approach. The most profound element in this aspect is the computation of the gravity effect of a certain mass distribution at random field points. This is crucial to potential field interpretation since its main goal is to compare and eventually correlate the observed with the calculated anomalies. Blakely (1996) describes these terms placing them in the overall concepts of forward and inverse modeling procedures. Observed is the anomaly that is measured on the field either through direct observations or through synthesis of the given potential harmonic coefficients of a known earth gravity model. It represents the observed gravity field at the corresponding

field point. Calculated is the corresponding quantity that can be evaluated mathematically by computing the gravitational effect of known underlying structures at the same point. The comparison between the calculated and the observed signal permits the initiation of an iterative loop which adapts gradually selected model parameters of the calculated signal (such as the geometry or consistency of the source) until both signals (calculated and observed) match adequately.

Governing quantity in the aforementioned evaluation of the calculated anomaly is the modeling choice of the underlying source. Despite the availability of known structures in the earth's interior, as is the case with recent global digital crustal databases, even small variations in the actual geometry and consistency of the anomalous source can affect the numerical value of the calculated anomaly considerably. In principle, the iterative scheme commences with some initial guess on the outline of the hidden source, usually with the shape of an idealized geometric body, to which a constant density is assigned throughout its volume in a first approximation. Different choices for the geometry of the ideal body may be linked to the nature of the problem or the available data, for example the choice of a right rectangular prism as the source volume element fits best the data implied by a digital elevation model in planar approximation.

For the computation of the gravity effect of the corresponding test bodies at arbitrary field points analytical, numerical or hybrid solutions may be elaborated. The present contribution deals with analytical solutions only, as they represent exact and rigorous expressions for the gravitational potential and its derivatives and are therefore superior to a numerical evaluation of the corresponding integral expressions. A vast literature exists on the topic of closed analytical expressions for the gravity effect of ideal bodies. Cylindrical mass distributions have received in proportion to other prismatic sources (prisms, polyhedra etc) the less attention, perhaps due to an existing restriction in terms of application to real case studies. Rosenbach (1947) dealt explicitly with analytical expressions for the gravitational effect of cylindrical mass distributions. Furthermore, the cylindrical shape received scientific scrutiny due to its central role in laboratory experiments with high sensitivity standards, as for example torsion balance measurements or gravitational constant experiments (Cook and Chen 1982; Chang 1988) and the proposal for the implementation of the STEP (Satellite Test of the Equivalence Principle) experiment in the 1990's (Lockerbie et al. 1995; 1996). The gravity effect of a conical mass distribution can be found only in selected references, mainly in connection with the gravity effect of a disc of finite thickness (Jung 1961; Talwani 1973). In the following, analytical expressions for the gravitational potential and attraction are presented for a computation point situated at the cone's symmetry axis.

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2. Potential and attraction of a right circular cone

The term *cone* is used to represent a three dimensional solid that is bounded by a planar face and the surface which is formed by the line segments joining every point of the perimeter of the base with a single point situated outside the basal plane, called the *apex*. When the surface is symmetric with respect to a line passing through the apex, then this line is called the *axis* of the cone. In the present contribution only right circular cones are considered. The term *right* is used to describe the fact that the axis passes through the center of the base and is perpendicular to the basal plane, while *circular* refers to the shape of the base, which in this case is a circle.

As far as the consistency of the attracting source is concerned, we deal here with right circular cones that define mass distributions of constant density ρ. Two geometrical distinctions with respect to the height-to-radius ratio of the conical distribution are performed. In the first case the scenario is examined where the radius of the circular base R equals the height of the cone h, i.e. the distance of the apex from the basal plane along the cone's axis. The second case represents the more general scenario, where $h \neq R$. In both cases analytical expressions for the potential and gravitational attraction are derived for a computation point situated at random points along the symmetry axis of the cone including its apex.

Radius of circular base equals height of cone (h=R)

Let us consider the conical mass distribution depicted in Figure 1. The cone is defined by a circular base of radius R and height $h=R$. The included mass is considered to be of constant density $\rho = ct$. The gravitational potential at a point P situated at a random point along the cone's symmetry axis besides its apex and outside the attracting masses, i.e. with coordinates $(0, 0, c)$ where $c > R$, will be given by the expression

$$V = G\rho \int_{z=0}^{R} \int_{s=0}^{r(z)} \int_{\alpha=0}^{2\pi} \frac{1}{l} \, du \qquad (1)$$

with $x = s \cos\alpha$, $y = s \sin\alpha$, $l = \sqrt{s^2 + (c-z)^2}$ and α expressing the angle between s and the x-axis on the $<x\,y>$ plane varying between zero and 2π (Figure 1). The finite volume element is defined as $du = dx\,dy\,dz = s\,ds\,d\alpha\,dz$. For the special case of Figure 1, where the height of the cone equals the radius of its base, the upper limit for parameter s will always be $R - z$, with z denoting the height of the horizontal plane passing through s from the conical basal plane.
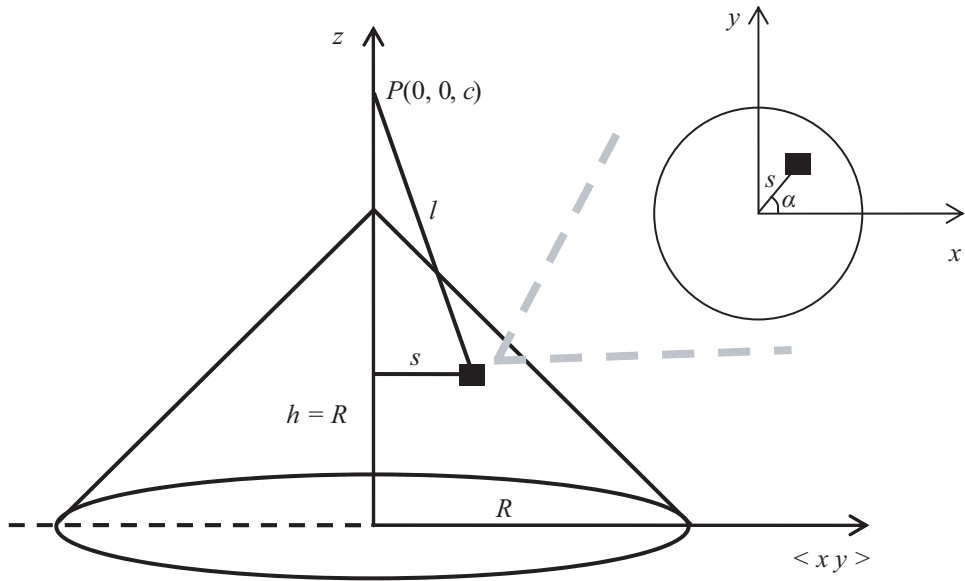
Thus, equation (1) becomes

Figure 1: *Geometric definitions linked to the special case of the circular cone, whose height equals the radius of its base. The computation point is situated at a random point along the symmetry axis of the attracting conical source with coordinates (0, 0, c).*

$$V = 2\pi G\rho \int\limits_{z=0}^{R} \int\limits_{s=0}^{R-z} \frac{s\,ds\,dz}{\sqrt{s^2 + (c-z)^2}} = 2\pi G\rho \int\limits_{z=0}^{R} \left[\sqrt{s^2 + (c-z)^2}\right]_0^{R-z} dz$$

$$= 2\pi G\rho \int\limits_{z=0}^{R} \left[\sqrt{(R-z)^2 + (c-z)^2} - c + z\right] dz = 2\pi G\rho\left(I_1 + I_2 + I_3\right). \tag{2}$$

The analytical solution of the individual integrals in (2) yields

$$I_2 = -c \int\limits_{z=0}^{R} dz = -c\left[z\right]_0^R = -c\,R \tag{3}$$

$$I_3 = \int\limits_{z=0}^{R} z\,dz = \left[\frac{1}{2}z^2\right]_0^R = \frac{1}{2}R^2 \tag{4}$$

$$I_1 = \int\limits_{z=0}^{R} \sqrt{R^2 + z^2 - 2Rz + c^2 + z^2 - 2cz}\ dz = \int\limits_{z=0}^{R} \sqrt{2z^2 - (2R+2c)z + (c^2 + R^2)}\ dz \tag{5}$$

Setting $A = 2$, $B = -(2R + 2c)$ and $C = c^2 + R^2$, the last integral of equation (5) has the form

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$$\int \sqrt{Az^2 + Bz + C}\ dz$$

and its analytical solution can be found from standard integral tables, e.g. Bronstein and Semendjajew (1996, pp 168-169) as

$$I_1 = \frac{2Az + B}{4A}\sqrt{Az^2 + Bz + C} + \frac{4AC - B^2}{8A}\frac{1}{\sqrt{A}}\ln\left[2\sqrt{A(Az^2 + Bz + C)} + 2Az + B\right]$$

Using this expression and the abbreviation $Q = 2z^2 - (2R + 2c)z + (c^2 + R^2)$ the closed solution for integral I_1 becomes

$$I_1 = \left[\frac{4z - 2R - 2c}{8}\sqrt{Q} + \frac{8(c^2 + R^2) - (-2R - 2c)^2}{16\sqrt{2}}\ln\left(2\sqrt{2Q} + 4z - 2R - 2c\right)\right]_0^R.$$

After a few intermediate steps of basic algebra we obtain

$$I_1 = -\frac{(c - R)^2}{4} + \frac{(c - R)^2}{4\sqrt{2}}\ln\left(2\sqrt{2}(c - R) + 2R - 2c\right)$$

$$+ \frac{R + c}{4}\sqrt{c^2 + R^2} - \frac{(c - R)^2}{4\sqrt{2}}\ln\left[2\sqrt{2}\sqrt{c^2 + R^2} - 2R - 2c\right]. \tag{6}$$

Inserting (3), (4) and (6) into (2) and performing some re-ordering the final expression for the gravitational potential can be obtained as

$$V = \pi G\rho\left[\frac{1}{2}(c - R)^2 - c^2 + \frac{(c - R)^2}{2\sqrt{2}}\ln\left(2\sqrt{2}(c - R) + 2R - 2c\right)\right.$$

$$\left. + \frac{R + c}{2}\sqrt{c^2 + R^2} - \frac{(c - R)^2}{2\sqrt{2}}\ln\left[2\sqrt{2}\sqrt{c^2 + R^2} - 2R - 2c\right]\right]. \tag{7}$$

The vertical derivative of equation (7) will provide the general expression for the gravitational attraction due to the conical mass distribution at the same point. This will read

$$V_z = \frac{\partial V}{\partial c} = G\rho\pi\left[-c - R + \frac{1}{2}\sqrt{c^2 + R^2} + \frac{1}{2}(R + c)\frac{c}{\sqrt{c^2 + R^2}}\right.$$

$$+ \frac{1}{\sqrt{2}}(c - R)\ln\left(2\sqrt{2}(c - R) - 2(c - R)\right)$$

$$\left. + \frac{1}{2\sqrt{2}}(c - R) - \frac{1}{\sqrt{2}}(c - R)\ln\left[2\sqrt{2}\sqrt{c^2 + R^2} - 2(R + c)\right]\right]$$

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$$-\frac{(c-R)^2}{2\sqrt{2}}\ \frac{\sqrt{2}\,c-\sqrt{c^2+R^2}}{\left[\sqrt{2}(c^2+R^2)-(R+c)\sqrt{c^2+R^2}\right]}\Bigg]. \tag{8}$$

For $c = R$, i.e. for a point situated at the apex of the cone, equation (8) simplifies to

$$V_z = G\rho\pi\left[-2R+\frac{1}{2}\sqrt{2}\,R+\frac{1}{\sqrt{2}}\,R\right]=G\rho\pi R\left[\sqrt{2}-2\right]. \tag{9}$$

Talwani (1973) follows a different approach than the direct integration of the corresponding kernel function using the geometric limits of the cone, which is discussed here. Integrating the expression for the gravity effect of a disc of radius r, which he derived independently, he obtains an analytical expression for the gravitational attraction for a point located at the symmetry axis of the cone as a function of angle β, which is the angle subtended by radius R, when viewed from the apex of the cone. Talwani's expression for the gravitational attraction computed at the apex of the cone reads (ibid., eq. 43)

$$V_z = 2\pi G\rho R(1-\cos\beta) \tag{10}$$

which is identical with expression (9) derived above, since for the case of the cone depicted in Figure 1 it holds $\beta = 45^{\circ}$, consequently $\cos\beta = \sqrt{2}/2$.

Radius of circular base is not equal to height of cone ($h \neq R$)

For the more general case, where the height of the cone h is not equal to the radius of its base R, the corresponding integral for the gravitational potential (eq. 2) becomes now

$$V = 2\pi G\rho \int\limits_{z=0}^{h}\ \int\limits_{s=0}^{R_z}\frac{s\,ds\,dz}{\sqrt{s^2+(c-z)^2}}\ . \tag{11}$$

All involved parameters follow the definitions given for the cone of Figure 1, while the computation point is again situated at $P(0, 0, c)$, with $c > h$, i.e. at the cone's symmetry axis but outside the attracting masses and not at the apex. Parameter R_z expresses the upper limit of parameter s and denotes the radius of the corresponding circle defined by the cross section of the horizontal plane and the conical distribution at z. Here, this parameter equals to

$$R_z = \frac{R(h-z)}{h}\ .$$

Integrating (11) with respect to s yields

$$V = 2\pi G\rho \int_{z=0}^{h} \left[\sqrt{s^2 + (c-z)^2} \right]_0^{R_z} dz = 2\pi G\rho \int_{z=0}^{h} \left[\sqrt{R_z^2 + (c-z)^2} - c + z \right] dz$$

$$= 2\pi G\rho \int_{z=0}^{h} \left(\sqrt{R^2 + \frac{R^2}{h^2}z^2 - 2Rz + c^2 + z^2 - 2cz} - c + z \right) dz$$

$$= 2\pi G\rho \int_{z=0}^{h} \left(\sqrt{\left(\frac{R^2}{h}+1\right)z^2 + (-2R - 2c)z + c^2 + R^2} - c + z \right) dz \qquad (12)$$

The last expression has the general form

$$V = 2\pi G\rho \int_{z=0}^{h} \left[\sqrt{Az^2 + Bz + C} - c - z \right] dz = 2\pi G\rho \left(I_1 + I_2 + I_3 \right) \qquad (13)$$

with

$$A = \frac{R^2}{h^2} + 1 \;,\;\; B = -2R - 2c \;\; \text{and} \;\; C = c^2 + R^2.$$

The individual solutions of the three definite integrals in (13) read respectively

$$I_2 = -c \int_{z=0}^{h} dz = -ch \qquad (14)$$

$$I_3 = \int_{z=0}^{h} z \, dz = \left[\frac{1}{2}z^2 \right]_0^h = \frac{1}{2}h^2 \qquad (15)$$

$$I_1 = \int_{z=0}^{h} \sqrt{Az^2 + Bz + C} \, dz$$

$$= \left[\frac{2Az + B}{4A} \sqrt{Az^2 + Bz + C} + \frac{4AC - B^2}{8A} \frac{1}{\sqrt{A}} \ln\left[2\sqrt{Az^2 + Bz + C} + 2Az + B \right] \right]_0^h$$

$$= \left[\frac{\left(2\left[\frac{R^2}{h^2}+1\right]z - 2R - 2c \right)}{4\left(\frac{R^2}{h^2}+1\right)} \sqrt{Az^2 + Bz + C} + \right.$$

$$+\frac{4\left(\dfrac{R^2}{h^2}+1\right)\left(c^2+R^2\right)-\left(-2R-2c\right)^2}{8\left(\dfrac{R^2}{h^2}+1\right)}\frac{1}{\sqrt{\dfrac{R^2}{h^2}+1}}\times$$

$$\times\ln\left(2\sqrt{A(Az^2+Bz+C)}+2\left(\frac{R^2}{h^2}+1\right)z-2R-2c\right)\Bigg]_0^h$$

$$=\frac{2Ah-2R-2c}{4A}\sqrt{Ah^2+Bh+C}+\frac{4AC-(2R+2c)^2}{8A}\frac{1}{\sqrt{A}}\times$$

$$\times\ln\left[2\sqrt{A(Ah^2+Bh+C)}+2Ah+B\right]-\frac{(-2R-2c)}{4A}\sqrt{C}-$$

$$-\frac{4AC-(2R+2c)^2}{8A}\frac{1}{\sqrt{A}}\ln\left(2\sqrt{AC}-2R-2c\right).\tag{16}$$

From equations (13) – (16) the final analytical expression for the gravitational potential can be obtained

$$V=2\pi G\rho\left[\frac{2Ah-2R-2c}{4A}\sqrt{Ah^2+Bh+C}+\right.$$

$$+\frac{4AC-(2R+2c)^2}{8A}\frac{1}{\sqrt{A}}\ln\left[2\sqrt{A(Ah^2+Bh+C)}+2Ah+B\right]-$$

$$\left.-\frac{(-2R-2c)}{4A}\sqrt{C}-\frac{4AC-(2R+2c)^2}{8A}\frac{1}{\sqrt{A}}\ln\left(2\sqrt{AC}-2R-2c\right)-ch+\frac{1}{2}h^2\right].$$

$$\tag{17}$$

The gravitational attraction can be obtained directly by differentiating the last equation with respect to c and taking into account the fact that $\partial A/\partial c=0$, $\partial B/\partial c=-2$ and $\partial C/\partial c=2c$.

3. Concluding remarks

Analytical expressions for the computation of the gravity effect of idealized attracting sources are a fundamental tool in gravity field modeling and interpretation. Known geometrical shapes can be used to approximate unknown disturbing sources or model the signal of known distributions. The areas of application of such formulas stretch from local or regional gravity field modeling and analysis to

the modeling of the gravity signal of known anomalous sources in the frame of high accuracy experiments in the laboratory. Although numerical integration techniques and high capacity personal computers permit the direct numerical evaluation of any multidimensional integral expression, analytical solutions offer mathematical elegance and geometrical insight into the underlying aspects and the nature of gravity field. The right circular conical mass distribution is a case study which has not received wide attention in the relevant geophysical and geodetic literature. Analytical expressions for the gravitational potential and attraction have been presented for a computation point situated at the cone's symmetry axis and outside the attracting masses. Two special cases were defined depending on the relation between the height of the cone and the radius of its circular base. For points outside the symmetry axis the analytical derivation of the corresponding expressions become more tedious and involve the use of elliptic integrals.

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